Nonadditivity of quantum capacities of quantum multiple-access channels and the butterfly network

2011 Phys. Scr. 84 045013

(http://iopscience.iop.org/1402-4896/84/4/045013)

IOP Publishing Physica Scripta
Phys. Scr. **84** (2011) 045013 (6pp) doi:10.1088/0031-8949/84/04/045013

Nonadditivity of quantum capacities of quantum multiple-access channels and the butterfly network

Peng Huang, Guangqiang He, Jun Zhu and Guihua Zeng

State Key Laboratory of Advanced Optical Communication Systems and Networks, Department of Electronic Engineering, Shanghai Jiaotong University, Shanghai 200240, People's Republic of China

E-mail: peakeagle1985@hotmail.com

Received 3 June 2011
Accepted for publication 30 August 2011
Published 22 September 2011
Online at stacks.iop.org/PhysScr/84/045013

Abstract
Multipartite quantum information transmission without additional classical resources is investigated. We show purely quantum superadditivity of quantum capacity regions of quantum memoryless multiple-access (MA) channels, which are not entanglement breaking. Also, we find that the superadditivity holds when the MA channel extends to the quantum butterfly network, which can achieve quantum network coding. The present widespread effects for the channels which enable entanglement distribution have not been revealed for multipartite scenarios.

PACS numbers: 03.67.Hk, 89.70.Kn

(Some figures in this article are in colour only in the electronic version.)

1. Introduction

The combination of quantum theory and classical information theory leads to an emerging field, namely, quantum information theory. One of the main problems of quantum information theory is how to obtain the capacity of a quantum channel, i.e. how much information can be transmitted down the quantum channels [1]. Unlike the classical channels, a quantum channel can be applied not only to transmit classical information [2, 3], but also to transmit quantum data [4, 5] and classical private information [6]. In particular, quantum capacity $Q(\mathcal{N})$ is the quantity that quantifies how large a Hilbert space of states the channel $\mathcal{N}$ can transmit asymptotically with vanishing errors, classical capacity $C(\mathcal{N})$ is the maximal rate of classical information the channel $\mathcal{N}$ can transmit asymptotically and faithfully and classical private capacity $P(\mathcal{N})$ quantifies the maximal rate of transmitted classical information that further should be inaccessible to the environment.

Very surprisingly, Smith and Yard [7] showed theoretically the superadditivity phenomena of quantum capacities for the bipartite scenario, in which two quantum channels with zero transmission capacity can have a nonzero capacity when they are used together. For the multipartite communication scenarios, i.e. channels with several senders and several receivers, the superadditivity of the quantum capacity has also been revealed in [8] when supported by free two-way classical communications. Recently, the superadditivity effects of quantum capacities are newly observed for the multiple-access (MA) entanglement breaking channels, i.e. channels that cannot create entanglement between the sender and the receiver, without additional resources [9]. On the other hand, nonadditivity of classical capacity has been found for the bipartite scenario in the product state encoding case [10], and for MA channels [9, 11] in an asymptotic way whether they are entanglement breaking channels or not, without additional resource support. The observation of superadditivities of classical capacities for the MA entanglement breaking channels demonstrates that there is a borderline between superadditivities of bipartite and multipartite systems, since the bipartite entanglement breaking channels cannot contribute such superadditivity effects [12–14]. However, purely superadditivity effects of quantum capacities for MA quantum channels that enable entanglement distribution, i.e. channels that can create entanglement between the sender and the receiver, have not yet been revealed when additional classical communications are excluded.

In this paper, we explore whether superadditivity effects still hold for the MA channels which can be used for entanglement distribution, since purely quantum superadditivity phenomena have not yet been revealed for the multipartite channels studied before when side resources are unavailable. To achieve this aim, we first construct a class of two general MA entanglement distribution channels. Then, importantly, we construct a special ideal three-access entanglement distribution channel, and extend the ideal three-access to a noisy version and a quantum butterfly network [15], which can achieve quantum network coding [15–18]. Generally, quantum network coding enables transmitting quantum information simultaneously, as well as cross transmitting, even when there is only one channel connecting the two sides in the butterfly network. We show strong nonadditivities of quantum capacities of the class of two general MA channels, the three-access channel and the butterfly network when assisted with highly entangling channels. Thus, it can be concluded that quantum capacity regions of MA channels without side resources are also nonadditive whether they are entanglement breaking or not. As provided in [9, 11], the additivity theorem of classical capacity for the quantum network violates the locality rule for the discrete classical networks that in any classical MA network primitive it is impossible to improve the transfer rate of one sender by adding resources to another sender. Here, we extend this additivity theorem classical capacity to the quantum capacity scenario for the quantum network.

This paper is organized as follows. In section 2, we give a brief introduction to the quantity we are investigating, namely the quantum capacity of the quantum channel, for the bipartite and multipartite scenarios. We then construct a class of general MA entanglement distribution channels, propose an ideal multi-access quantum channel and a noisy version of it and show the superadditivity effects of the quantum capacity regions of these two channels without additional resources in section 3. In section 4, we demonstrate that the quantum capacity region of the quantum butterfly network, which can achieve quantum network coding, is also nonadditive when no additional resources are involved. Finally, the conclusions drawn are presented in section 5.

2. Quantum capacity of a quantum channel

Quantum communication channels that can be physically pictured as transmissions of quantum systems from sender to receiver can be used to transfer classical or quantum information. For the transmission of classical information, the classical bits are first encoded in quantum states, which are then transmitted via a quantum channel, whereas in the case of quantum information, unknown quantum states are directly encoded and transmitted between the communicators. Mathematically, a quantum channel is represented by a completely positive, trace-preserving linear map $\mathcal{N}$, which maps from $\mathcal{B}(\mathcal{H}_A)$ to $\mathcal{B}(\mathcal{H}_B)$, where $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on the space $\mathcal{H}$, and $\mathcal{H}_A$ and $\mathcal{H}_B$ are the input and output Hilbert spaces. Given n uses of a quantum channel $\mathcal{N}:\mathcal{B}(\mathcal{H}_A)\to\mathcal{B}(\mathcal{H}_B)$, what we would like to find is a quantum code $\mathcal{C}_n\subset\mathcal{H}_{A^{\otimes n}}$ and a decoding operation $\mathcal{D}_n:\mathcal{B}(\mathcal{H}_{B^{\otimes n}})\to\mathcal{B}(\mathcal{C}_n)$, such that every state $|\psi\rangle\in\mathcal{C}_n$ can be decoded with high fidelity after being sent through the channel $\mathcal{N}^{\otimes n}$, that is, $\mathcal{D}_n\circ\mathcal{N}^{\otimes n}(|\psi\rangle\langle\psi|)\approx|\psi\rangle\langle\psi|$. The rate of the code $\mathcal{C}_n$, which is defined as

$$R=\frac{1}{n}\log\dim\mathcal{C}_n, \tag{1}$$

quantifies the amount of quantum information transmitted between the sender and the receiver. More formally, the rate R can be achieved if for all $\epsilon>0$ and sufficiently large n, there is a code $\mathcal{C}_n$ and a decoding operation $\mathcal{D}_n$, where $1/n\log\dim\mathcal{C}_n\geqslant R$ for $\mathcal{C}_n\subset\mathcal{H}_{A^{\otimes n}}$ and $\mathcal{D}_n:\mathcal{B}(\mathcal{H}_{B^{\otimes n}})\to\mathcal{B}(\mathcal{C}_n)$ such that for all $|\psi\rangle\in\mathcal{C}_n$, the fidelity

$$F(|\psi\rangle,\mathcal{D}_n\circ\mathcal{N}^{\otimes n}(|\psi\rangle\langle\psi|))\geqslant 1-\epsilon. \tag{2}$$

The quantum capacity $Q(\mathcal{N})$ of the bipartite quantum channel $\mathcal{N}$ is defined as the supremum of all such achievable rates.

The definition of quantum capacity can be generalized to an MA channel, i.e. a channel has many spatially separated senders $A_1,\ldots,A_n$ and one receiver B. Now the input Hilbert space is given by the tensor product of the Hilbert spaces of the senders, $\mathcal{H}_A=\mathcal{H}_{A_1}\otimes\ldots\mathcal{H}_{A_n}$, while the channel is also $\mathcal{N}:\mathcal{B}(\mathcal{H}_A)\to\mathcal{B}(\mathcal{H}_B)$. For an MA quantum channel and any subset $\tilde{A}$ of the senders, one can define a rate $R_{\tilde{A}\to B}$ as the measurement of the amount of quantum information sent from the set of senders $\tilde{A}$ to receiver B, and a rate vector $\mathbf{R}=(R_{A_1\to B},\ldots,R_{A_j\to B},\ldots,R_{A_n\to B})$ to evaluate the quantum capacity region. Obviously, we have $\mathcal{H}_{\tilde{A}}\cong\mathcal{H}_B$, where $\mathcal{H}_{\tilde{A}}=\bigotimes_{A_i\in\tilde{A}}\mathcal{H}_{A_i}$ is the Hilbert space of parties in the subset $\tilde{A}$ [8], and $\sum_i R_{A_i\to B}\leqslant R_{\tilde{A}\to B}$ for $A_i\in\tilde{A}$ because of the possible superadditivity of the quantum capacity of quantum channels [7, 9].

It should be noted that the parties in the complementary set of $\tilde{A}$ may prepare their system in some suitable state to achieve the multiparty communication. Considering all the varieties of quantum codes and decoding operations for each choice of subset of the senders, i.e. all the optimal achievable rates over all possible choices of $\tilde{A}$, the quantum capacity region of the MA channel is the largest space expanded by the rate vector $\mathbf{R}$.

3. Superadditivity of the quantum capacity of an MA channel

First of all, we construct a class of two general MA entanglement distribution channels via known bipartite quantum channels that violate the additivity of quantum capacities. For bipartite channels, the effects have been found for two channels with zero quantum capacities [7], namely the Horodecki channel $\mathcal{N}_{\mathrm{H}}$ with nonzero private capacity and the 50% erasure channel $\mathcal{N}_A$ with zero private capacity. It is shown that the quantum capacity $Q(\mathcal{N}_{\mathrm{H}}\times\mathcal{N}_A)>0$. For the bipartite channel $\mathcal{N}$ with nonzero quantum capacity, it can be deduced from [7] that these superadditivity effects only exist under the condition of $Q(\mathcal{N})<0.5P(\mathcal{N})$. However, recent studies on the nonadditivity of private capacity [19, 20] show that this condition in the bipartite case is actually not necessary. Therefore, MA quantum channels that admit nonadditivity of quantum capacities can be easily constructed from the bipartite channels as follows.

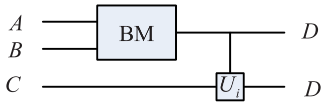

Figure 1. The MA teleportation channel. BM denotes the Bell measurement and U_i is the controlled unitary operator $U_i \in \{I, \sigma_x, \sigma_z, \mathrm{i}\sigma_y\}$, where σ_i is the Pauli matrix.

Suppose two bipartite entanglement distribution channels that admit the nonadditivity of the quantum capacity to be $\mathcal{N}_1$ and $\mathcal{N}_2$. Then the first MA channel with two senders A, B and a receiver C is between A and C is the channel $\mathcal{N}_1$ and between B and C is the identity channel $\mathcal{N}^{\mathrm{Id}}_{B\to C}$; the second MA channel can be constructed similarly with $\mathcal{N}_2$ replacing $\mathcal{N}_1$. Thus, it can be easily seen that the two MA entanglement distribution channels admit nonadditivity of the whole quantum capacity regions, and all the bipartite entanglement distribution channels that admit nonadditivity of quantum capacities can be used to construct the MA channels violating the additivity. However, are there any two MA entanglement distribution channels in which the constitutive bipartite channels cannot violate nonadditivity of quantum capacities, but also admit nonadditivity?

In the following, we show superadditivity of quantum capacity regions for the entanglement distribution MA channel, in which the bipartite channels cannot violate nonadditivity of quantum capacities, when combined with a highly entangling channel. The MA channel is plotted in figure 1; we call it here the MA teleportation channel $\mathcal{N}^{\mathrm{T}}$. Alice, Bob and Charlie have two-dimensional (2D) inputs, while David has a 2D and a 4D output. The channel carries out Bell measurement on two qubits sent by Alice and Bob and transmits a measurement result to David. Simultaneously, the channel transforms the qubit sent from Charlie to David depending on the measurement outcomes. The outcomes $|\Phi^+\rangle_D$, $|\Phi^-\rangle_D$, $|\Psi^+\rangle_D$ and $|\Psi^-\rangle_D$ correspond to unitary operations I, σ_z, σ_x and $\mathrm{i}\sigma_y$, respectively.

Considering the single use of the channel $\mathcal{N}^{\mathrm{T}}$, Alice (Bob) can teleport an unknown qubit state to David when Bob (Alice) and Charlie send a pair of maximally entangled qubit states. Thus, the transmission rate between Alice (Bob) and Charlie can be obtained as $R_{A(B)} = 1$ (R_i denotes the rate $R_{i\to C}$ with $i = A, B$ for short in this section) according to equation (1). This gives rise to rate vectors $(R_A, R_B, R_C) = (1, 0, 0)$ and $(R_A, R_B, R_C) = (0, 1, 0)$. Also, the channel can transmit a qubit from Charlie to David, while Alice and Bob send a fixed maximally entangled state. This corresponds to the rate vector $(R_A, R_B, R_C) = (0, 0, 1)$. We now can easily obtain the quantum capacity and classical private capacity of the transmission channel between a single sender i and receiver D as $Q(\mathcal{N}^{\mathrm{T}}_i) = P(\mathcal{N}^{\mathrm{T}}_i) = 1$ for $i = A, B$ and C. Since the channel carries out the complete von Neumann measurement on two qubits, the rate $R_A + R_B + R_C$ cannot be greater than 1. It should be mentioned that the input of entangled signals across the same channels will not increase the extra rate, the quantum capacity of the channel $\mathcal{N}^{\mathrm{T}}$ itself admits a single-letter formula. Hence, the quantum capacity region of $\mathcal{N}^{\mathrm{T}}$ is given by $Q(\mathcal{N}^{\mathrm{T}}) = \{(R_A, R_B, R_C) : R_A + R_B + R_C \leqslant 1\}$.

The other channel is the identity qubit channel $\mathcal{N}^{\mathrm{Id}}_{A\to D}$ which transmits single qubits from Alice to David faithfully. Its quantum capacity region can be easily obtained as $Q(\mathcal{N}^{\mathrm{Id}}_{A\to D}) = \{(R_A, R_B, R_C) : R_A \leqslant 1, R_B = 0, R_C = 0\}$.

Now we find the quantum capacity region $Q(\mathcal{N}^{\mathrm{T}} \otimes \mathcal{N}^{\mathrm{Id}}_{A\to D})$. Also, it can be easily demonstrated that $Q(\mathcal{N}^{\mathrm{T}} \otimes \mathcal{N}^{\mathrm{Id}}_{A\to D})$ admits a single-letter formula. Firstly, we consider the senders who transmit quantum information alone. For Alice, she can optimize her rate by teleporting a qubit through channel $\mathcal{N}^{\mathrm{T}}$ and transmitting another qubit through channel $\mathcal{N}^{\mathrm{Id}}_{A\to D}$. This gives the rate vector $(R_A, R_B, R_C) = (2, 0, 0)$. For the other two senders, the rate vectors $(R_A, R_B, R_C) = (0, 1, 0)$ and $(R_A, R_B, R_C) = (0, 0, 1)$ can also be achieved. Secondly, we consider two of the senders who cooperate to send quantum messages. It can be easily seen that the rate $R_A + R_B \leqslant 2$ and $R_A + R_C \leqslant 2$ can be achieved, since the channel $\mathcal{N}^{\mathrm{Id}}$ can transmit one qubit from Alice to David, and channel $\mathcal{N}^{\mathrm{T}}$ can teleport a qubit from Bob to David or transmit one qubit from Charlie to David. Now we turn our attention to evaluating the rate $R_B + R_C$. When Alice sends half of the maximally entangled pair of qubits through the channel $\mathcal{N}^{\mathrm{Id}}_{A\to D}$ and the other half through the channel $\mathcal{N}^{\mathrm{T}}$, Bob and Charlie can transmit two qubits to David. Consider the general state transmitted by Bob and Charlie $|\psi\rangle_{BC} = a|00\rangle_{BC} + b|01\rangle_{BC} + c|10\rangle_{BC} + d|11\rangle_{BC}$; we get

$$\begin{aligned}|\Phi^+\rangle_{AA'} \otimes |\psi\rangle_{BC} = {} & \tfrac{1}{2}|\Phi^+\rangle_{AB} \otimes |\psi\rangle_{A'C} + \tfrac{1}{2}|\Phi^-\rangle_{AB} \\ & \otimes \sigma_z^C \sigma_z^{A'} \sigma_z^C |\psi\rangle_{A'C} + \tfrac{1}{2}|\Psi^+\rangle_{AB} \\ & \otimes \sigma_x^C \sigma_x^{A'} \sigma_x^C |\psi\rangle_{A'C} + \tfrac{1}{2}|\Psi^-\rangle_{AB} \\ & \otimes (\mathrm{i}\sigma_y^C)(\mathrm{i}\sigma_y^{A'})(\mathrm{i}\sigma_y^C)|\psi\rangle_{A'C}, \qquad (3)\end{aligned}$$

where the subscript A' denotes the input of the channel $\mathcal{N}^{\mathrm{Id}}_{A\to D}$. It can be seen from equation (3) that David can perform two modificatory unitary operations on the received two qubits of the two 2D outputs according to the Bell measurement results of the 4D output, and then receive two qubits faithfully. For instance, if the Bell measurement result is $|\Psi^+\rangle_D$, then David uses the Pauli matrix σ_x. Hence, the rate $R_B + R_C \leqslant 2$ can also be achieved. Lastly, the rate $R_A + R_B + R_C$ cannot be greater than 2 because of the complete von Neumann measurement on two qubits in the channel $\mathcal{N}^{\mathrm{T}}$. Therefore, the capacity region $Q(\mathcal{N}^{\mathrm{T}} \otimes \mathcal{N}^{\mathrm{Id}}_{A\to D})$ is given by

$$R_A + R_B + R_C \leqslant 2, \quad R_B \leqslant 1, \quad R_C \leqslant 1. \qquad (4)$$

It can be clearly seen that the quantum capacity region of the product of the MA teleportation channel and the identity channel $Q(\mathcal{N}^{\mathrm{T}} \otimes \mathcal{N}^{\mathrm{Id}}_{A\to D})$ is greater than the geometric sum of two quantum capacity regions of the two individual channels, i.e. $Q(\mathcal{N}^{\mathrm{T}})$ and $Q(\mathcal{N}^{\mathrm{Id}}_{A\to D})$, which are depicted in figure 2. Hence, we have shown the nonadditivity of quantum capacity regions of the MA channels without additional resources even when the single bipartite channels $\mathcal{N}^{\mathrm{T}}_i$ cannot violate nonadditivity when assisting with the channel $\mathcal{N}^{\mathrm{Id}}_{A\to D}$ for $i = A, B$ and C. This simple example illustrates that the nonadditivity effect of quantum capacity regions of different channels can also occur naturally for the MA channels even when they are not entanglement breaking channels, and it shows that superadditivity effects of the quantum

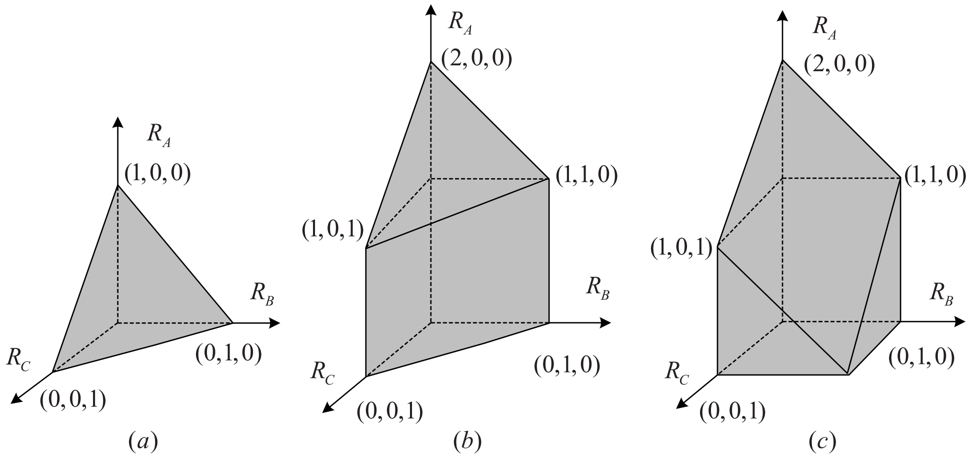

Figure 2. (a) Quantum capacity region of the channel $\mathcal{N}^{\mathrm{T}}$; (b) geometric sum of quantum capacity regions of channels $\mathcal{N}^{\mathrm{Id}}_{A\to D}$ and $\mathcal{N}^{\mathrm{T}}$; (c) quantum capacity region of the product of two channels $\mathcal{N}^{\mathrm{Id}}_{A\to D}$ and $\mathcal{N}^{\mathrm{T}}$.

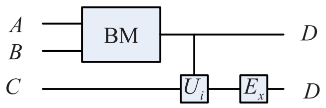

Figure 3. The noisy version of the MA teleportation channel. BM denotes the Bell measurement, and U_i is the controlled unitary operator $U_i \in \{I, \sigma_x, \sigma_z\}, \mathrm{i}\sigma_y$, where σ_i is the Pauli matrix. E_x is the 2D quantum erasure channel.

capacity regions can occur for a larger range of entanglement distribution channels.

It is worth noting that the superadditivity of quantum capacity regions effect also exists when the MA teleportation channel extends to a noisy version. We denote the noisy MA teleportation channel by $\mathcal{N}^{\mathrm{T}}_{E_x}$, which is depicted in figure 3. The channel $\mathcal{N}^{\mathrm{T}}_{E_x}$ acts as a combination of $\mathcal{N}^{\mathrm{T}}$ and the 2D quantum erasure channel E_x [21], which transmits the input state faithfully with probability $1-x$, and with probability x replaces the input by a unique state, i.e. an erasure symbol, orthogonal to all the input states. For $0 < x < 0.5$, the quantum erasure channel, and hence also the noisy MA channel $\mathcal{N}^{\mathrm{T}}_{E_x}$, can achieve quantum distribution. The quantum capacity region is given by $Q(\mathcal{N}^{\mathrm{T}}_{E_x}) = \{(R_A, R_B, R_C) : R_A + R_B + R_C \leqslant Q\}$, where Q is the quantum capacity of the quantum erasure channel E_x with the formula $Q = 1 - 2x$ for $0 < x < 0.5$ [22]. The other channel is the identity qubit channel from Charlie to David with the quantum capacity region $Q(\mathcal{N}^{\mathrm{Id}}_{C\to D}) = \{(R_A, R_B, R_C) : R_A = 0, R_B = 0, R_C \leqslant 1\}$.

Now we consider the quantum capacity region of the tensor product of these two channels. Assuming Alice (Bob) sends half of the maximally entangled pair of qubits through the channel $\mathcal{N}^{\mathrm{T}}_{E_x}$ and Charlie sends the other half through the channel $\mathcal{N}^{\mathrm{Id}}_{C\to D}$, the product channel can teleport a qubit from Bob (Alice) to David when he performs a modificatory operation on the received qubit according to the received Bell measurement results. Thus, we get $(R_A, R_B, R_C) = (1, 0, 0)$ and $(R_A, R_B, R_C) = (0, 1, 0)$. Also, Charlie can optimize his transmission by sending qubit states through the two individual channels. This gives the rate vector $(R_A, R_B, R_C) = (0, 0, 1 + Q)$. On the other hand, there is $R_A + R_B + R_C \leqslant 1 + Q$, since the total rate cannot be greater than the quantum capacity of the product of the 2D quantum erasure channel and the identity qubit channel. Hence, three extreme points of the quantum capacity region of the product channel are given by

$$\begin{aligned} (R_A, R_B, R_C) &= (1, 0, 0), \\ (R_A, R_B, R_C) &= (0, 1, 0), \\ (R_A, R_B, R_C) &= (0, 0, 1 + Q). \end{aligned} \tag{5}$$

These extreme points prove the superadditivity of these quantum capacity regions, since $Q < 1$ for $0 < x < 0.5$. Also, we can easily obtain the quantum capacity of the transmission channel between the single sender i and receiver D as $Q(\mathcal{N}^{\mathrm{T}}_{E_x,i}) = Q = 1 - 2x$ for $i = A, B$ and C, and the single bipartite channels $\mathcal{N}^{\mathrm{T}}_{E_x,i}$ also admit additivity when assisting with the channel $\mathcal{N}^{\mathrm{Id}}_{A\to D}$. Supposing that the classical private capacity of the transmission channel between the single sender i and receiver D is valued as the maximum $P(\mathcal{N}^{\mathrm{T}}_{E_x,i}) = C(\mathcal{N}^{\mathrm{T}}_{E_x,i}) = 1 - x$ for $i = A, B$ and C, the condition $Q(\mathcal{N}_i) < 0.5P(\mathcal{N}_i)$ can be transformed to $x > \frac{1}{3}$. Obviously, both the MA channels $\mathcal{N}^{\mathrm{T}}$ and $\mathcal{N}^{\mathrm{T}}_{E_x}$ break the condition of producing superadditivity effects of the quantum capacities for the bipartite entanglement distribution channels in [7], since the effects for $\mathcal{N}^{\mathrm{T}}_{E_x}$ always exist for $0 < x < 0.5$.

4. Superadditivity of quantum capacity of a quantum butterfly network

In this section, we consider the quantum capacity region of a quantum butterfly network described in figure 4. The butterfly network can be seen as a quantum channel $\mathcal{N}^B$ with six senders and two receivers, which is composed of two noisy MA teleportation channels $\mathcal{N}^{\mathrm{T}}_{E_x}$ with modifications. Each sender has a 2D quantum input, and each receiver has a 2D quantum output and a 4D classical one. The node T transforms the Bell measurement results $|\Phi^+\rangle_D$, $|\Phi^-\rangle_D$, $|\Psi^+\rangle_D$ and $|\Psi^-\rangle_D$ to 4D classical bits 00, 01, 10 and 11 correspondingly and takes a binary addition.

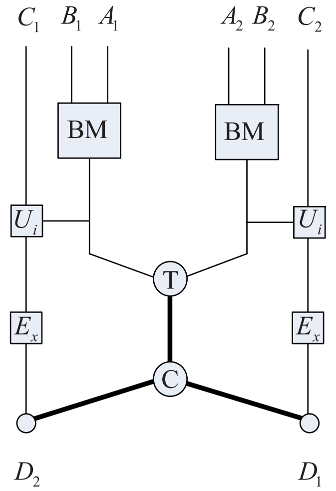

Figure 4. The quantum butterfly network $\mathcal{N}^{\mathrm{B}}$, which enables perfect quantum network coding. The thick (thin) lines stand for classical (quantum) channels. BM denotes the Bell measurement, and U_i is the controlled unitary operator $U_i \in \{I, \sigma_x, \sigma_z, \mathrm{i}\sigma_y\}$, where σ_i is the Pauli matrix. E_x is the 2D quantum erasure channel.

The node C copies the input classical bits and transfers them to two sides. Since here we consider the quantum information transmission, the intact unknown qubits can be only transmitted to one receiver, i.e. D_1 or D_2. We denote the total rate vector as $\mathbf{R} = \{\mathbf{R}^{\mathrm{P}}, \mathbf{R}^{\mathrm{C}}\}$, where $\mathbf{R}^{\mathrm{P}} = (R_{A_1\to D_2}, R_{B_1\to D_2}, R_{C_1\to D_2}, R_{A_2\to D_1}, R_{B_2\to D_1}, R_{C_2\to D_1})$ and $\mathbf{R}^{\mathrm{C}} = (R_{A_1\to D_1}, R_{B_1\to D_1}, R_{C_1\to D_1}, R_{A_2\to D_2}, R_{B_2\to D_2}, R_{C_2\to D_2})$ are called the parallel rate vector and cross rate vector here, respectively.

We first consider parallel transmission. When A_1, B_1 (A_2, B_2) send a fixed pair of maximally entangled qubits, the channel can transmit Q qubits from C_1 (C_2) to D_2 (D_1), and teleport another Q qubits from A_2 (A_1) or B_2 (B_1) to D_1 (D_2). Furthermore, when two pairs of the senders A_1, B_1 and A_2, B_2 transmit two pairs of fixed maximally entangled qubits, Q qubit transmission from C_1 to D_2 and from C_2 to D_1 can also be achieved. On the other hand, the total rate of parallel transmission cannot be greater than $2Q$. Hence, the extreme points of the parallel transmission rate vector are given by

$$\begin{aligned} &\mathbf{R}^{\mathrm{P}} = (Q,0,0,0,0,0), && \mathbf{R}^{\mathrm{P}} = (0,Q,0,0,0,0),\\ &\mathbf{R}^{\mathrm{P}} = (0,0,Q,0,0,0), && \mathbf{R}^{\mathrm{P}} = (0,0,0,Q,0,0),\\ &\mathbf{R}^{\mathrm{P}} = (0,0,0,0,Q,0), && \mathbf{R}^{\mathrm{P}} = (0,0,0,0,0,Q). \end{aligned} \tag{6}$$

Also, the channel $\mathcal{N}^{B}$ can fulfil quantum network coding by using the protocol proposed in [18]. In particular, when $A_1(B_1)$, C_2 and $A_2(B_2)$, C_1 send two pairs of maximally entangled qubit states, $B_1(A_1)$ and $B_2(A_2)$ can simultaneously transmit Q qubits to D_1 and D_2, respectively. It should be noted that the receivers should modify the received qubits with unitary operations according to the received bits, where 00, 01, 10 and 11 correspond to Pauli matrices I, σ_z, σ_x and $\mathrm{i}\sigma_y$, respectively. Thus, we obtain the six extreme points of the cross transmission rate vector as

$$\begin{aligned} &\mathbf{R}^{\mathrm{C}} = (Q,0,0,0,0,0), && \mathbf{R}^{\mathrm{C}} = (0,Q,0,0,0,0),\\ &\mathbf{R}^{\mathrm{C}} = (0,0,Q,0,0,0), && \mathbf{R}^{\mathrm{C}} = (0,0,0,Q,0,0),\\ &\mathbf{R}^{\mathrm{C}} = (0,0,0,0,0,0), && \mathbf{R}^{\mathrm{C}} = (0,0,0,0,0,0). \end{aligned} \tag{7}$$

Now we take into account the case when the channel $\mathcal{N}^{\mathrm{B}}$ is assisted by a product of two identity channels, $\mathcal{N}^{\mathrm{Id}}_{C_1C_2\to D_2D_1} = \mathcal{N}^{\mathrm{Id}}_{C_1\to D_2} \otimes \mathcal{N}^{\mathrm{Id}}_{C_2\to D_1}$. Obviously, its quantum capacity region is given by

$$R_{C_1\to D_2} \leqslant 1, \quad R_{C_2\to D_1} \leqslant 1. \tag{8}$$

Also, we first consider the parallel transmission rate vector. When A_2, B_2 (A_1, B_1) send fixed maximally entangled pairs, A_1 (A_2) or B_1 (B_2) can teleport qubit information to D_2 (D_1) through the identity channel $\mathcal{N}^{\mathrm{Id}}_{C_1\to D_2}$ ($\mathcal{N}^{\mathrm{Id}}_{C_2\to D_1}$), and C_1 (C_2) can also transmit Q qubits to D_2 (D_1) through channel $\mathcal{N}^{B}$, when performing modificatory unitary operations on the received qubits according to the received bits. Furthermore, C_1 and C_2 can at most transmit $1+Q$ qubits to D_2 and D_1, when A_1, B_1 and A_2, B_2 send fixed maximally entangled pairs. Thus, the extreme points of the parallel transmission rate vector of the product channel are given by

$$\begin{aligned} &\mathbf{R}^{\mathrm{P}} = (1,0,0,0,0,0), && \mathbf{R}^{\mathrm{P}} = (0,1,0,0,0,0),\\ &\mathbf{R}^{\mathrm{P}} = (0,0,1,0,0,0), && \mathbf{R}^{\mathrm{P}} = (0,0,0,1,0,0),\\ &\mathbf{R}^{\mathrm{P}} = (0,0,0,0,1+Q,0), && \mathbf{R}^{\mathrm{P}} = (0,0,0,0,0,1+Q). \end{aligned} \tag{9}$$

For cross transmission, when A_2, B_2 send fixed maximally entangled pairs, and A_1 sends half of the maximally entangled pair of qubits through the channel $\mathcal{N}^{\mathrm{B}}$ while C_2 sends the other half through channel $\mathcal{N}^{\mathrm{Id}}_{C_2\to D_1}$, B_1 can teleport one qubit to D_1 with a modificatory operation. Also, A_2, B_1 and B_2 can cross transmit one qubit in the same way. Hence, we obtain the extreme points of the cross transmission rate vector of the product channel as

$$\begin{aligned} &\mathbf{R}^{\mathrm{C}} = (1,0,0,0,0,0), && \mathbf{R}^{\mathrm{C}} = (0,1,0,0,0,0),\\ &\mathbf{R}^{\mathrm{C}} = (0,0,1,0,0,0), && \mathbf{R}^{\mathrm{C}} = (0,0,0,1,0,0),\\ &\mathbf{R}^{\mathrm{C}} = (0,0,0,0,0,0), && \mathbf{R}^{\mathrm{C}} = (0,0,0,0,0,0). \end{aligned} \tag{10}$$

These extreme points in equations (6)–(10) prove the nonadditivities of the quantum capacity regions for both parallel and cross transmissions. Hence, we have shown the nonadditivity of the quantum capacity region of the quantum butterfly network. It should be mentioned that the sum of the rates from A_1 or B_1 to D_1 and from A_2 or B_2 to D_2 of the product channel remain unchanged at $2Q$, i.e. the assistance of identity channels cannot improve the simultaneous cross transmission, since D_1, D_2 still cannot confirm exactly which unitary operation should be used to recover the received qubits. However, the assistance of highly entangling channels can help improve the single cross transmission rate, i.e. achieving one qubit, which results in the superadditivity of the quantum capacity region for cross transmission. It can be seen from the superadditivity of quantum capacity regions of the entanglement distributing MA channel and network that even the quantum channels, which are unrelated to the senders, can be applied to assist in improving their transmissions of quantum information. This is a natural phenomenon coming from quantum teleportation and does not exist in classical communications. In particular, it is significant for quantum network communication, since the higher rate of quantum information transmission implies a higher rate of entanglement distribution, which has broad applications in quantum communication.

5. Conclusion

We have introduced the definition of the quantum capacity for the bipartite quantum channels and extended it to MA quantum channels. Then we have constructed a simple class of general MA entanglement distribution channels that admit nonadditivity of quantum capacities via two bipartite channels that violate the nonadditivity of quantum capacities. Furthermore, we construct two MA entanglement distribution channels each with three senders and one receiver, i.e. an ideal MA channel and a noisy one, in which the bipartite channels cannot violate nonadditivity of quantum capacities, and a quantum butterfly network which can achieve a quantum coding network. We have evaluated in detail the quantum capacity regions of the two three-access quantum channels and the butterfly network, and quantum capacities when they are assisted by ideal identity channels without additional classical communications in an asymptotic way. We have shown the superadditivity of quantum capacity regions of the ideal MA channel geometrically, and proved that the superadditivity effects still hold for the noisy MA channel and the quantum butterfly network. The present effects for entanglement distributing channels, to some extent, discriminate the three-access channel with other MA quantum channels, since these naturally widespread effects have never been found in other multipartite scenarios when no additional resources are involved. Hence, we can conclude that nonadditivities also hold for quantum capacities of the MA channels whether they are entanglement breaking or not, without additional resource support.

Acknowledgments

We thank the referees for valuable comments on the manuscript. This work was supported by the National Natural Science Foundation of China (grant numbers 60970109 and 60801051).

References

[1] Bennett C H and Shor P W 2004 *Science* **303** 1784
[2] Holevo A S 1996 *IEEE Trans. Inform. Theory* **44** 269
[3] Schumacher B and Westmoreland M D 1997 *Phys. Rev.* A **56** 131
[4] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 *Phys. Rev.* A **54** 3824
[5] Barnum H, Nielsen M A and Schumacher B 1998 *Phys. Rev.* A **57** 4153
[6] Bennett C H and Brassard G 1984 *Proc. IEEE Int. Conf. on Computers, System and Signal Processing* (New York: IEEE) pp 175–9
[7] Smith G and Yard J 2008 *Science* **321** 1812
[8] Dür W, Cirac J I and Horodecki P 2004 *Phys. Rev. Lett.* **93** 020503
[9] Grudka A and Horodecki P 2010 *Phys. Rev.* A **81** 060305
[10] Hastings M B 2009 *Nature Phys.* **5** 255
[11] Czekaj L and Horodecki P 2009 *Phys. Rev. Lett.* **102** 110505
[12] Horodecki M, Shor P W and Ruskai M B 2003 *Phys. Math. Phys.* **15** 629
[13] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 *Phys. Rev.* A **54** 3824
[14] Shor P W 2002 *J. Math. Phys.* **43** 4334
[15] Leung D, Oppenheim J and Winter A 2006 arXiv:quant-ph/0608223
[16] Hayashi M, Iwama K, Nishimura H, Raymond R and Yamashita S 2007 *STACS 2007: 24th Int. Symp. on Theoretical Aspects of Computer Science* (*Lecture Notes in Computer Science* vol 4393) (Berlin: Springer) p 610
[17] Iwama K, Nishimura H, Raymond R and Yamashita S 2006 arXiv:quant-ph/0611039
[18] Hayashi M 2007 *Phys. Rev.* A **76** 040301
[19] Li K, Winter A, Zou X B and Guo G C 2009 *Phys. Rev. Lett.* **103** 120501
[20] Smith G and Smolin J A 2009 *Phys. Rev. Lett.* **103** 120503
[21] Grassl M, Beth Th and Pellizzari T 1997 *Phys. Rev.* A **56** 33
[22] Bennett C H, DiVincenzo D P and Smolin J A 1997 *Phys. Rev. Lett.* **78** 3217