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Quantum capacity of Pauli channels with memory

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Abstract
The amount of coherent quantum information that can be reliably transmitted down the memory Pauli channels with Markovian correlated noise is investigated. Two methods for evaluating the quantum capacity of the memory Pauli channels are proposed to try to trace the memory effect on the transmissions of quantum information. We show that the evaluation of quantum capacity can be reduced to the calculation of the initial memory state of each successive transmission. Furthermore, we derive quantum capacities of the memory phase flip channel, bit flip channel and bit-phase flip channel. Also, a lower bound of the quantum capacity of the memory depolarizing channel is obtained. An increase of the degree of memory of the channels has a positive effect on the increase of their quantum capacities.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Unlike the classical channels, quantum channels have several distinct capacities, among which the quantum capacity of quantum channels, i.e. the amount of coherent quantum information that can be reliably transmitted per channel use, is one of the peculiar characteristics of quantum channels [1]. Significant attention has so far been paid to determining the exact quantum capacities of the memoryless quantum channels [1–8], i.e. channels in which the noise acts independently on each channel use. However, only the quantum capacities of some very special memoryless quantum channels are known, such as the amplitude-damping [6], phase flip [7] and erasure [8] channels, and most others are not known. So, there are several amendatory capacities that are defined [1, 4] to evaluate the upper and lower bounds of those quantum capacities.

In fact, the assumption of having the noise acting on successive channel uses independently is unrealistic, since correlations between errors in real-world applications of noisy quantum channel are common. Recently, a great deal of achievements [9–15] have been made regarding the classical and quantum capacities of memory quantum channels, i.e. the ones with correlated noise. In particular, quantum capacities of the so-called forgetful channels, for which the memory effects decay exponentially with time, have been studied [10, 14, 16], and coding theorems for their quantum capacities have been proved [14]. In classical information theory, one type of memory channel with known capacity is the one with Markovian correlated noise. Bowen and Mancini [9] extended this classical memory channel to quantum information theory and considered the quantum channel with Markovian correlated noise, where such a correlated noise quantum channel is modeled with the use of unitary operations between the transmitted states, an environment state and a memory state with a finite dimension.

We evaluate the amount of coherent quantum information that can be reliably transmitted down the Pauli quantum channels with memory, and trace the memory effect acting on the quantum information transmission. The correlations between the errors are considered temporally over the successive channel uses. In fact, they can also be considered spatially between the uses of the parallel channels equivalently for a consideration of computation of the capacity. We find it is possible to evaluate the quantum capacities of the Pauli channels by using the model including a memory state. It should be remarked that the dimension of the channel memory state is determined by the number of Kraus operators in the single channel expansion and the correlation length of the channel. In particular, we derive the quantum capacities of the memory phase flip channel, bit flip channel and bit-phase flip channel and also obtain a lower bound of the quantum capacity of the depolarizing channel with memory. Moreover, we explore the effects of the memory caused by the Markovian correlated noise on these capacities.

This paper is organized as follows. In section 2, we give a detailed introduction to the quantity we are investigating, namely the quantum capacity, and then a brief introduction to the forgetful channels. In section 3, we construct the finite-memory channel with Markovian corrected noise, and explore the quantum capacity of the memory Pauli channels based on this model. In particular, we consider some special Pauli channels with memory, including the memory phase flip channel, bit flip channel, bit-phase flip channel and depolarizing channels. Finally, our conclusions are presented in section 4.

2. Quantum capacity of quantum channels

Quantum communication channels that can be physically pictured as transmissions of quantum systems from the sender to the receiver can be used to transfer classical or quantum information. For transmission of classical information, the classical bits are firstly encoded in quantum states, which are then transmitted via a quantum channel. For the case of quantum information, unknown quantum states are directly encoded and transmitted between the communicators. One of the fundamental tasks in quantum information theory is to derive the capacity of a quantum channel for transmitting the classical or quantum information, i.e. the maximum classical or quantum information that a quantum channel can transmit per channel use with vanishing errors. Specifically, the quantum capacity can be quantified as the maximum Hilbert space of states that the quantum channel can transmit asymptotically and faithfully.

Mathematically, a quantum channel is represented by a completely positive, trace-preserving (CPTP) linear map $\mathcal{N}$, which maps from $\mathcal{B}(\mathcal{H}_1)$ to $\mathcal{B}(\mathcal{H}_2)$, where $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on the space $\mathcal{H}$ and $\mathcal{H}_1$ and $\mathcal{H}_2$ are the input and output Hilbert space. According to the Stinespring dilation theorem [17], a quantum channel (or the CPTP map) $\mathcal{N}$ can be always described by an isometric map U from the input Hilbert space A to the combined Hilbert space of the output B and the environment Hilbert space E, followed by a partial trace over E. It can be explicitly represented in the form

$$\mathcal{N}(\rho) = \mathrm{Tr}_E U\rho U^\dagger, \tag{1}$$

where U satisfies $U^\dagger U = I_A$. Also, due to the Kraus representation theorem [18], any quantum channel with input space A and output space B can be expressed as

$$\mathcal{N}(\rho) = \sum_k A_k \rho A_k^\dagger, \tag{2}$$

where A_k are linear maps from A to B with $\sum_k A_k^\dagger A_K = I_B$. The natural complementary channel [19–21] $\tilde{\mathcal{N}}$ can then be defined by taking the partial trace over B so that

$$\tilde{\mathcal{N}}(\rho) = \mathrm{Tr}_B U\rho U^\dagger. \tag{3}$$

Thus, the coherent information [22] of a quantum channel $\mathcal{N}$ with a reference state ρ can be expressed as

$$I_{\mathrm{C}}(\rho, \mathcal{N}) = S(\mathcal{N}(\rho)) - S(\tilde{\mathcal{N}}(\rho)), \tag{4}$$

where $S(\rho) = -\mathrm{Tr}\,\rho \log \rho$ is the von Neumann entropy. The quantum capacity of the memoryless channel $\mathcal{N}$ is given by

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} I_{\mathrm{C}}(\mathcal{N}^{\otimes n}), \tag{5}$$

where $I_{\mathrm{C}}(\mathcal{N}) = \max_\rho I_{\mathrm{C}}(\rho, \mathcal{N})$, and $\mathcal{N}^{\otimes n}$ denotes n uses of a quantum channel. When a channel satisfies the additivity condition $I_{\mathrm{C}}(\mathcal{N}^{\otimes n}) = n I_{\mathrm{C}}(\mathcal{N})$, the quantum capacity can be simplified to a 'single-letter' formula $Q(\mathcal{N}) = I_{\mathrm{C}}(\mathcal{N})$. Most classes of the quantum channels cannot satisfy the additivity condition except the degradable channels shown in [19].

3. Finite-memory quantum channel

The unitary interaction expression for a quantum channel introduced above provides an intuitive understanding of the open quantum systems, besides providing a method for calculating the quantum capacity. For memoryless channels, the transmitted information and noise sources are treated as independent random variables, whereas for real-world noisy quantum channels, this independence should be removed since the correlations between the errors are common. Actually, the noise correlations are sometimes necessary for certain quantum communication [23]. This corresponds to a physical example of memory quantum channels. We will introduce a model with finite-memory state proposed in [9] for the class of correlated noise channels. Then we explore the amount of coherent quantum information for the quantum channels with correlated noise effects.

3.1. Construction of the finite-memory channel

We first consider the memoryless channel with input state given by $\rho_A = \sum_i \lambda_i \rho_i$. A quantum channel is described as a CPTP linear map, which may be represented as a unitary operation on the enlarged input vector space including the initial input state ρ and a known environment state, which is initially in a pure state $\rho_E = |0\rangle_E\langle 0|$. The output state of the single use of channel is given by

$$\mathcal{N}(\rho_A) = \mathrm{Tr}_E \left[U_{AE}(\rho_A \otimes \rho_E) U_{AE}^\dagger \right]. \tag{6}$$

When a sequence of quantum states is transmitted then

$$\mathcal{N}(\rho_A^n) = \mathrm{Tr}_E \left[U_{A_n E_n} \cdots U_{A_1 E_1} (\rho_A^n \otimes \rho_E^n) U_{A_1 E_1}^\dagger \cdots U_{A_n E_n}^\dagger \right], \tag{7}$$

where the state ρ_A^n denotes n input states for the n uses of quantum channel, which may be entangled, and $\rho_E^n = |\phi_E\rangle\langle\phi_E|$ is the environment state with $|\phi_E\rangle = |0_{E_1}\rangle \cdots |0_{E_n}\rangle$. The trace over the environment means over all the environment states. The output can be expressed also from the Kraus decomposition as

$$\mathcal{N}(\rho_A^n) = \sum_{j_1\cdots j_n} (A_{j_n} \otimes \cdots \otimes A_{j_1}) \rho_A^n \left(A_{j_1}^\dagger \otimes \cdots \otimes A_{j_n}^\dagger \right). \tag{8}$$

For the memory channels, one model is constructed that the transmitted states going through the channel act with a unitary interaction on the same channel memory state but independent environment states. Therefore, this memory channel model includes a memory state and n independent environment states for n uses of channels. We obtain

$$\mathcal{N}(\rho_A^n) = \mathrm{Tr}_{ME}\Big[U_{A_nME_n}\cdots U_{A_1ME_1}(\rho_A^n\otimes|M\rangle\langle M|\otimes\rho_E^n) \times U^\dagger_{A_1ME_1}\cdots U^\dagger_{A_nME_n}\Big]. \tag{9}$$

The quantum channel of which the unitary $U_{A_iME_i}$ can be factorized into $U_{A_iE_i}U_M$ or $U_{A_iM}U_{E_i}$ corresponds to a memoryless channel or a perfect memory channel, since the memory or the environment is traced out. Now the Kraus decomposition is given by

$$\mathcal{N}(\rho_A^n) = \mathrm{Tr}_M\sum_{j_1\cdots j_n}(A_{j_nM}\cdots A_{j_1M})(\rho_A^n\otimes|M\rangle\langle M|) \times \left(A^\dagger_{j_1M}\cdots A^\dagger_{j_nM}\right). \tag{10}$$

It should be mentioned that the unitary operation in equation (10) may not be factorized as a product of operators like in the form of equation (8). Since there is the same qubit memory state for n uses of quantum channel, the transmission of the quantum channel will affect the output of the next transmission, which corresponds to the memory effect of this channel. Actually, the application of the same channel memory state makes the n uses of the quantum channel correlated.

3.2. The forgetful memory channel

It is known that equation (5) gives only an upper bound for a generic memory channel. Only in some cases has it been proved that it is the true quantum capacity, for example for a forgetful channel [10, 14, 16]. Forgetful channels are defined in [14], in which the environment is modeled in two parts: the memoryless one and the one inducing memory effects. A direct feature of forgetful channels is that the memory effects decrease exponentially with time. So in some cases, with permitted error, the forgetful channel can be mapped into a memoryless one. This can be clarified by the double-blocking strategy [10, 14] as shown in the following. Considering blocks of $N+L$ uses of the channel $\mathcal{N}$, we actually code and decode for the first N uses and ignore the remaining L idle uses. The output can be expressed as $\mathcal{N}_{N+L}(\rho)$ when it acts on the state ρ on the space $\mathcal{H}^{\otimes N}$. When considering the M uses of such blocks, the corresponding output of $\mathcal{N}_{M(N+L)}$ can be approximated to $(\mathcal{N}_{N+L})^{\otimes M}(\rho)$. This property can be explained as follows [10, 14]:

$$\|\mathcal{N}_{M(N+L)}(\rho) - (\mathcal{N}_{N+L})^{\otimes M}(\rho)\|_1 \leqslant k(M-1)c^L, \tag{11}$$

where $c<1$, $\|X\|_1 = \mathrm{Tr}(\sqrt{X^\dagger X})$ is the trace norm and k is a constant only depending on the memory model. Equation (11) states that the error induced by the replacement of the corresponding memoryless channel decays to zero exponentially fast with the number L of idle uses in a single block. The intuitional explanation may be that the correlations among different blocks dies out during the idle uses. Also, equation (11) is a sufficient condition to prove coding theorems for forgetful quantum memory channels. Thus, the system satisfying inequality (11) can be seen as a forgetful channel in the following.

3.3. Quantum capacity of memory Pauli channels

For n uses of a Pauli channel, a general situation can be represented by Kraus operators in the following form:

$$A_{j_1\cdots j_n} = \sqrt{p_{j_1\cdots j_n}}\sigma_{j_1}\cdots\sigma_{j_n}, \tag{12}$$

where $\sum_{j_1\cdots j_n}p_{j_1\cdots j_n}=1$ and $\sigma_{j_1}\cdots\sigma_{j_n}$ is a sequence of unitary operators with $\sigma_0 = I$, and $\sigma_1,\sigma_2,\sigma_3$ correspond to the Pauli matrices. For the memoryless channel, $p_{j_1\cdots j_n} = p_{j_1}p_{j_2}\cdots p_{j_n}$.

We consider the class of memory channels with Markovian correlated noise, that is, $p_{j_n|j_{n-1}j_{n-2}\cdots j_k} = p_{j_n|j_{n-1}}$ for all $k<n$. Thus, $p_{j_1\cdots j_n}$ can be reduced as

$$p_{j_1\cdots j_n} = p_{j_1}p_{j_2|j_1}\cdots p_{j_n|j_{n-1}}, \tag{13}$$

where $p_{j_n|j_{n-1}}$ is the conditional probability that the operator σ_{j_n} is applied to the nth qubit given that the operator $\sigma_{j_{n-1}}$ was applied to the $(n-1)$th qubit. We assume that $p_{j_n|j_{n-1}} = (1-\mu)p_{j_n} + \mu\delta_{j_n,j_{n-1}}$, where μ is the correlation parameter corresponding to the memory effects of the quantum channel. This means that with probability μ the same operator is applied to both of the qubits, while with probability $1-\mu$ the two operators are uncorrelated. According to equation (10), the output of n uses of the memory Pauli channel with Markovian correlated noise is given by

$$\mathcal{N}(\rho_A^n) = \sum_{j_1\cdots j_n}p_{j_n|j_{n-1}}p_{j_{n-1}|j_{n-2}}\cdots p_{j_2|j_1}p_{j_1}(\sigma_{j_n}\otimes\cdots\otimes\sigma_{j_1}) \times \rho_A^n(\sigma^\dagger_{i_1}\otimes\ldots\otimes\sigma^\dagger_{j_n}), \tag{14}$$

where the set σ_{j_k} are the identity matrix and the Pauli matrices for the single use of the channel on the kth qubit for $j_k = 0, 1, 2, 3$. Also, the unitary operator in equation (9) can be expressed as

$$U_{A_iME_i}|\phi_A^{(i)}\rangle|j_M\rangle|0_{E_i}\rangle = \sum_k\sqrt{p_{k|j}}\sigma_k^{(i)}|\phi_A^{(i)}\rangle|k_M\rangle|j_{E_i}\rangle, \tag{15}$$

where $|\phi^{(i)}\rangle$ is the input state and $|j_M\rangle$ is the initial memory state. We consider general input message state ρ_A^n and general memory input state $\rho_M = \sum_{j_1l_1}\gamma_{j_1l_1}|j_1\rangle\langle l_1|_M$ with $\sum_{j_1}\gamma_{j_1j_1}=1$, in which the value $\gamma_{j_1j_1}$ is determined by the value of the probability vector of the channel error operators $P=(p_0, p_1, \ldots, p_m)$. The maximum amount of coherent quantum information for n uses of the quantum channel is given by

$$I_C(\mathcal{N}) = \max_{\rho_A^n}\left[S\left(\rho_M^B\right) - S\left(\rho_M^E\right)\right], \tag{16}$$

where

$$\rho_M^B = \mathcal{N}(\rho_A^n) = \sum_{j_1}\gamma_{j_1j_1}\sum_{j_2\cdots j_{n+1}}p_{j_{n+1}|j_n}\cdots p_{j_2|j_1} \times\left(\sigma^{(n)}_{j_{n+1}}\otimes\cdots\otimes\sigma^{(1)}_{j_2}\right) \times\rho_A^n\left(\sigma^{(1)\dagger}_{j_2}\otimes\cdots\otimes\sigma^{(n)\dagger}_{j_{n+1}}\right) \tag{17}$$

and

$$\rho_M^E = \tilde{\mathcal{N}}(\rho_A^n) = \sum_{j_1} \gamma_{j_1 j_1} \sum_{j_2\cdots j_{n+1}} p_{j_{n+1}|j_n} \cdots p_{j_2|j_1} \rho_E, \quad (18)$$

with $\rho_E = |j_n^{(n)}\rangle \otimes \cdots \otimes |j_1^{(1)}\rangle\langle j_1^{(1)}| \otimes \cdots \otimes \langle j_n^{(n)}|_E$. The operator $\sigma_{j_k}^{(i)}$ denotes the action on the ith qubit for $j_k = 0, 1, 2, 3$, and $|j_k^{(i)}\rangle_E$ is the ith environment state of the whole environment by n uses of the quantum channel.

It can be seen that the output of the n uses of the channel in equation (17) is consistent with the one in equation (14). This verifies the equivalence of the two representations of the memory quantum channel. We can conclude from the above computation that the outputs of the channel and the complementary channel are only related with the diagonal elements of the input memory state, that is, $\rho_M^{\text{eff}} = \sum_{j_1} \gamma_{j_1 j_1} |j_1\rangle\langle j_1|_M$. By using the strong convexity of trace norm and Kolmogorov distance [10], we can directly prove that the condition in equation (11) is fulfilled for this finite-memory model by equation (14), and we obtain

$$\|\mathcal{N}_{M(N+L)}(\rho) - (\mathcal{N}_{N+L})^{\otimes M}(\rho)\|_1 \leqslant M\mu^{L+1}. \quad (19)$$

Thus, the finite-memory model for Pauli channels is forgetful and the computation from equation (5) gives the true quantum capacity. According to the subadditive property of von Neumann entropy, the quantum capacity of this class of memory Pauli channels with Markovian correlated noise is additive. Since the value of the quantum capacity is only positive related to $S(\rho_A^n)$ when the initial memory state has predetermined, it achieves the maximum when the n input product states $S(\rho_A^n)$ are the product state of n maximally mixed states of $\rho_A = \sum_{i=0}^{d-1} \frac{1}{d}|i\rangle\langle i|_A$, where d is the dimension of the channel.

3.4. Special Pauli channels

In this subsection, we explore the quantum capacity of some special Pauli channels with Markovian correlated noise based on the above computation, and calculate the quantum capacity in another way, so as to try to trace the evolution of the memory effect on transmissions. Firstly, consider the phase flip channel with Markovian correlated noise, and the output state of n uses of the channel can be derived directly from equation (14), where the set σ_{j_k} is defined as $\sigma_0 = I$, $\sigma_1 = \sigma_z$ and $p_0 = 1-p$, $p_1 = p$ for $j_k = 0, 1$, respectively. The maximal coherent quantum information for transmitting n qubits can be easily calculated from equation (16) as

$$I_{\mathrm{C}}(\mathcal{N}_z) = n - H(p) - (n-1)[(1-p)H(p_{0|0}) + pH(p_{1|1})], \quad (20)$$

where $p_{0|0} = (1-\mu)(1-p)+\mu$, $p_{1|1} = (1-\mu)p+\mu$, and $H(p) = -p\log_2 p - (1-p)\log_2(1-p)$ is binary Shannon entropy. Also, the coherent quantum information of transmission of n qubits quantum information can be seen as n times of the coherent quantum information of a single transmission of one qubit information because of the additivity of the von Neumann entropy for product input states. For the first transmission of one qubit, the output density operator of the memory phase flip channel is given by

$$\begin{aligned}\mathcal{N}_z^{(1)}(\rho_A) &= \sum_{j_1} \gamma_{j_1 j_1} \sum_{j_2} p_{j_2|j_1} \sigma_{j_2} \rho_A \sigma_{j_2}^{\dagger} \\ &= \gamma_{00}\left(p_{0|0}\sigma_0\rho_A\sigma_0^{\dagger} + p_{1|0}\sigma_1\rho_A\sigma_1^{\dagger}\right) \\ &\quad + \gamma_{11}\left(p_{0|1}\sigma_0\rho_A\sigma_0^{\dagger} + p_{1|1}\sigma_1\rho_A\sigma_1^{\dagger}\right), \quad (21)\end{aligned}$$

where $p_{1|0} = 1 - p_{0|0}$, $p_{0|1} = 1 - p_{1|1}$, and $\gamma_{00} = 1-p$, $\gamma_{11} = p$ are the squares of the amplitudes for the initial memory state determined by the values of the initial probability vector of the channel error operators. Similarly, the output density operator of the complementary memory phase flip channel can be derived as

$$\begin{aligned}\tilde{\mathcal{N}}_z^{(1)}(\rho_A) &= \sum_{j_1} \gamma_{j_1 j_1} \sum_{j_2} p_{j_2|j_1} |j_1\rangle\langle j_1|_E \\ &= \gamma_{00}|0\rangle\langle 0|_E + \gamma_{11}|1\rangle\langle 1|_E. \quad (22)\end{aligned}$$

Therefore, when the input message state is the maximally mixed state with the form $\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$, the coherent quantum information of the memory phase flip channel for transmitting the first qubit achieves its maximum as

$$I_{\mathrm{C}}^{(1)}(\mathcal{N}_z) = S(\mathcal{N}_z^{(1)}(\rho_A)) - S(\tilde{\mathcal{N}}_z^{(1)}(\rho_A)) = 1 - H(p), \quad (23)$$

which is directly related with the value p corresponding to the squares of the amplitudes of the initial memory state. Now we consider the second transmission of the qubit. It can be seen that the maximization of the coherent quantum information only refers to the input message state, which is independent of the output of the complementary channel. So, we can also choose the maximally mixed state as the input message state to evaluate the quantum capacity. According to equation (15), the initial memory state for the second transmission is changed to $\rho_M^0 = p_{0|0}|0\rangle\langle 0|_M + p_{1|0}|1\rangle\langle 1|_M$ with the probability $1-p$ and $\rho_M^1 = p_{0|1}|0\rangle\langle 0|_M + p_{1|1}|1\rangle\langle 1|_M$ with probability p. Thus, we obtain $\mathcal{N}_z^{(2)}(\rho_A) = \sum_{j_2} \gamma_{j_2 j_2}^0 \sum_{j_3} p_{j_3|j_2} \sigma_{j_3} \rho_A \sigma_{j_3}^{\dagger}$, $\tilde{\mathcal{N}}_z^{(2)}(\rho_A) = \sum_{j_2} \gamma_{j_2 j_2}^0 \sum_{j_3} p_{j_3|j_2} |j_2\rangle\langle j_2|_E$ with probability $1-p$, and also can obtain $\mathcal{N}_z^{(2)}(\rho_A) = \sum_{j_2} \gamma_{j_2 j_2}^1 \sum_{j_3} p_{j_3|j_2} \sigma_{j_3} \rho_A \sigma_{j_3}^{\dagger}$, $\tilde{\mathcal{N}}_z^{(2)}(\rho_A) = \sum_{j_2} \gamma_{j_2 j_2}^1 \sum_{j_3} p_{j_3|j_2} |j_2\rangle\langle j_2|_E$ with probability p, where $\gamma_{ii}^j = p_{i|j}$. It follows that the maximal coherent quantum information of the memory phase flip channel for transmitting the second qubit takes the form

$$I_{\mathrm{C}}^{(2)}(\mathcal{N}_z) = 1 - (1-p)H(p_{0|0}) - pH(p_{1|1}). \quad (24)$$

After the second transmission, the memory state remains unchanged because the probability to be the state ρ_M^0 is $(1-p)p_{0|0} + pp_{0|1} = 1-p$, and the probability to stay ρ_M^1 is $(1-p)p_{1|0} + pp_{1|1} = p$. Also, the memory state will also remain unchanged for the rest of the transmissions. Therefore, the maximum value of coherent quantum information for each transmission of the rest of $n-2$ qubits is $I_{\mathrm{C}}^{(i)}(\mathcal{N}_z) = I_{\mathrm{C}}^{(2)}(\mathcal{N}_z)$

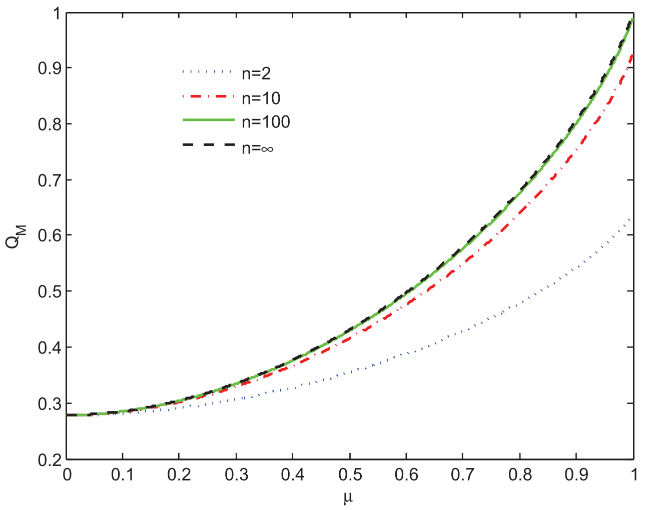

Figure 1. Quantum capacity Q_M as a function of the degree of memory of the channel for $p = 0.8$.

for $3 \leqslant i \leqslant n$. Now we obtain

$$\sum_{i=1}^{n} I_C^{(i)}(\mathcal{N}_z) = n - H(p) - (n-1)[(1-p)H(p_{0|0}) + pH(p_{1|1})], \tag{25}$$

which is consistent with the result in equation (20). The quantum capacity can be derived in the form

$$Q_M(\mathcal{N}_z) = \lim_{n\to\infty} \frac{1}{n}\{n - H(p) - (n-1)[(1-p)H(p_{0|0}) + pH(p_{1|1})]\}. \tag{26}$$

This expression is the same as the outcome proposed in [10], in which the memory phase flip channel is modeled as a Markov chain. In our computation model, the introduction of memory state makes the calculation of quantum capacity more explicit and we can deduce the concrete output density operator after each transmission. It can be seen that computation of the initial memory states is the key point to obtain the quantum capacities. When the operator set in equation (14) or (21) is defined as $\sigma_0 = I$, $\sigma_1 = \sigma_x$ or $\sigma_0 = I$, $\sigma_1 = \sigma_y$ for $j_k = 0, 1$, the description of the model changes to the bit flip channel or bit-phase flip channel. From equation (16) or equations (21) and (22), we can directly derive the quantum capacities of these two memory Pauli channels as

$$Q_M(\mathcal{N}_x) = Q_M(\mathcal{N}_y) = Q_M(\mathcal{N}_z) = Q_M. \tag{27}$$

In particular, while $\mu = 0$, the memory channels become memoryless, and the quantum capacities of the memory phase flip, bit flip and bit-phase flip all become $Q_M = 1 - H(p)$, and the quantum capacity $Q_M = 1$ for $\mu = 1$, which means that the perfect memory channels are asymptotically noiseless. The relationship between the quantum capacity Q_M and the degree of memory of the channel is plotted in figure 1 while $n = 2, 10, 100, \infty$. It shows that the quantum capacity increases with increasing degree of memory of the channel. In the following, we consider the depolarizing channel with Markovian correlated noise.

For a depolarizing channel, the set of Kraus operators σ_{j_k} in equation (13) are defined as $\sigma_{00} = I, \sigma_{01} = \sigma_x, \sigma_{10} = \sigma_y, \sigma_{11} = \sigma_z$, with corresponding probability $p_{00} = 1 - p$, $p_{01} = \frac{p}{3}$, $p_{10} = \frac{p}{3}$, $p_{11} = \frac{p}{3}$ for $j_k = 00, 01, 10, 11$, respectively. However, for the two-dimensional depolarizing channel, one qubit effective input memory state $\rho_M^{\mathrm{eff}} = \sum_{j_1} \gamma_{j_1 j_1} |j_1\rangle\langle j_1|_M$ cannot match the description of the unitary of the memory channel in equation (15), since the dimension of the input memory state should be at least larger than the number of Kraus operators for a single use of the channel. Thus, we expand the memory state to four dimensions with two qubits to match the four Kraus operators σ_{j_k}, but without changing the dimension of the environment. We can rewrite equation (15) as

$$U_{A_i M E_i} |\phi_A^{(i)}\rangle |j_M\rangle |0_{E_i}\rangle = \sum_k \sqrt{p_{k|j}}\sigma_k^{(i)} |k_M\rangle |j'_{E_i}\rangle, \tag{28}$$

where $j' = j \bmod 2$. The coherent quantum information of this memory depolarizing channel can be derived also from equation (16) with the maximally mixed state as the input state. Since have expanded the dimension of the memory, inevitably the amount of information transmitted to the environment is added according to equation (18). Thus, what we calculate from the above model stated in equation (28) is a lower bound of the quantum capacity of the depolarizing channel with Markovian correlated noise. This lower bound can be derived directly from equations (5) and (16). Here we consider the second way, that is, to calculate the coherent quantum information for each transmission of one qubit for n qubits of quantum information. For the first transmission, the effective initial memory state is now given by

$$\rho_M^{\mathrm{eff}} = \sum_{j_1} \gamma_{j_1 j_1} |j_1\rangle\langle j_1|_M = \mathrm{diag}(\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}), \tag{29}$$

where $\gamma_{ij} = p_{ij}$ for $i, j = 0, 1$. From equations (21) and (22), we obtain

$$I_C^{(1)}(\mathcal{N}_d) = 1 - H(p) - p\log_2 3. \tag{30}$$

After the first transmission, the density operator of the memory state is replaced by ρ_M^{00}, ρ_M^{01}, ρ_M^{10} and ρ_M^{11} with probability $1-p$, $\frac{p}{3}$, $\frac{p}{3}$, $\frac{p}{3}$, respectively, where

$$\begin{aligned}
\rho_M^{00} &= \mathrm{diag}(p_{00|00}, p_{01|00}, p_{10|00}, p_{11|00}),\\
\rho_M^{01} &= \mathrm{diag}(p_{00|01}, p_{01|01}, p_{10|01}, p_{11|01}),\\
\rho_M^{10} &= \mathrm{diag}(p_{00|10}, p_{01|10}, p_{10|10}, p_{11|10}),\\
\rho_M^{11} &= \mathrm{diag}(p_{00|11}, p_{01|11}, p_{10|11}, p_{11|11}).
\end{aligned} \tag{31}$$

Similar to the calculation of the phase flip channel, we obtain

$$\begin{aligned}
I_C^{(2)}(\mathcal{N}_d) &= (1-p)S\left(\rho_M^{00}\right) + \frac{p}{3}\left[S\left(\rho_M^{01}\right) + S\left(\rho_M^{10}\right) + S\left(\rho_M^{11}\right)\right]\\
&= 1 + r_0(1-p)\log_2 r_0 + r_1 p \log_2 r_1\\
&\quad + r_2(3-p)\log_2 r_2 + r_3 \log_2 r_3,
\end{aligned} \tag{32}$$

where $r_0 = (1-p)(1-\mu) + \mu$, $r_1 = (1-p)(1-\mu)$, $r_2 = \frac{p}{3}(1-\mu)$ and $r_3 = \frac{p}{3}(1-\mu) + \mu$. Actually, we can derive that the maximal coherent information for other

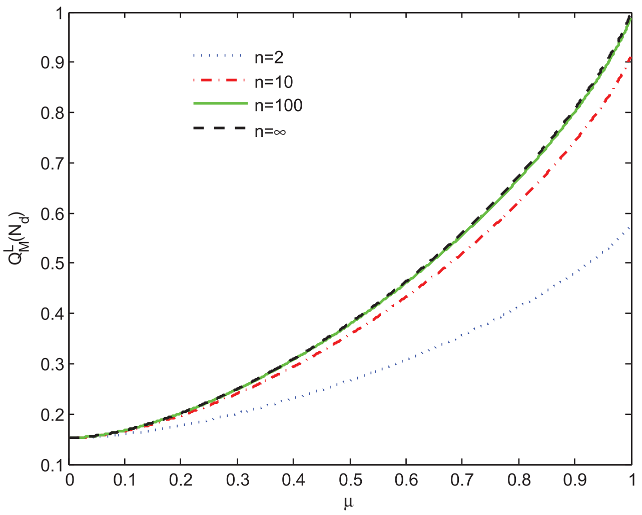

Figure 2. The lower bound of quantum capacity of the depolarizing channel $Q^L_M(\mathcal{N}_d)$ as a function of the degree of memory of the channel for $p = 0.15$.

transmissions are the same as $I_C^{(2)}$, which yields the lower bound of the quantum capacity of the memory depolarizing channel as

$$Q^L_M(\mathcal{N}_d) = \lim_{n\to\infty} \frac{1}{n}\left[(n-1)I_C^{(2)} + I_C^{(1)}\right]. \tag{33}$$

The relationship between the lower bound of the quantum capacity of the depolarizing channel with Markovian correlated noise $Q^L_M(\mathcal{N}_d)$ and the degree of memory of the channel μ is depicted in figure 2. It can be found that the capacity of this memory depolarizing channel increases with increasing the degree of memory. We would like to point out that for $\mu = 0$, the lower bound of the quantum capacity reduces to the hashing lower bound [24] of the memoryless depolarizing channel $Q^L_d = 1 - H(p) - p\log_2 3$. Moreover, the quantum capacity is maximized for the perfect memory channel. Unlike the classical capacity, of which the entangled input states may achieve a higher value than the product input states for a proper degree of memory of the channel [25]. The quantum capacities of the Pauli channels are always maximized by the product input states for the different definitions. It should be mentioned that for a general memory Pauli channel $\mathcal{N}(\rho_A) = \sum_{i=0}^{3} p_i\sigma_i\rho_A\sigma_i^\dagger$ with more than two Kraus operators, the quantum capacity can be only lower bounded by our computation model, since the quantum channel itself is physically limited to the two-dimensional space. In fact, even for the memoryless depolarizing channel, the exact quantum capacity formula of it has not been found until now, and only a lower bound [24] and a tight upper bound [4] are known.

4. Conclusion

A physical model including a memory state of finite dimension for a class of quantum communication channels with Markovian correlated noise has been investigated. Based on this model, we have calculated the coherent quantum information of Pauli channels with memory, which has been shown to be maximized by the separable maximally mixed input states. Then, the quantum capacities of these channels have been evaluated based on the model with two new methods. In particular, we have derived the quantum capacities of the memory phase flip channel, bit flip channel and bit-phase flip channel when considering the number of channel uses $n \to \infty$. However, for the general Pauli channels, the existence of four Kraus operators cannot match the two-dimensional memory physically defined in the Pauli channel model. So, we expanded the dimension of the memory to four and remodeled the unitary operator of the memory channel in order to evaluate the quantum capacity. In particular, we have evaluated the quantum capacity of the memory depolarizing channel and first derived a lower bound of it, which is consistent with the known hashing lower bound for $\mu = 0$. Actually, due to the limitation of the model, we could only obtain the lower bound of the quantum capacity of the Pauli channel with more than two Kraus operators. It should be mentioned that all the calculated quantum capacities increase with increasing memory of the quantum channels, and reach the maximum for the perfect memory ($\mu = 1$), which implies that the channels are asymptotically noiseless.

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