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Optimal Control of Continuous Variable Quantum Dense Coding Under Bosonic Structured Environments

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Abstract In this paper, how the dynamics of continuous variable entanglement affects quantum dense coding is quantitatively investigated. The spectrum of environment is Ohmic-like spectrum. The analytical expression of the mutual information is based on the assumption of weak system-reservoir interaction without Markovian and rotating wave approximation. By plotting the mutual information in terms of environment parameters such as reservoir temperature and spectral density, it's found that the mutual information of dense coding is monotonically decreasing function for Markovian interaction in Ohmic-like environments, while it oscillates for non-Markovian ones. Besides, it's suggested to select proper transmittance of beam-splitter to improve the performance of dense coding.

Keywords Quantum dense coding · Entanglement · Mutual information

1 Introduction

Entanglement is a quantum physics phenomenon, which has important applications in quantum communication [1]. Quantum dense coding [2], is one of the important applications of quantum communication, which has been demonstrated in experiment in discrete variable [3]. It can transmit two bits of classical information by only one qubit of quantum

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information with the help of entanglement. Recently, quantum dense coding with continuous variable has been proposed in theory [4] and demonstrated in experiment [5]. Later, a quantum controlled dense coding scheme with bright tripartite entangled states comes up [6] and demonstrated in experiment [7].

However, whether the continuous variable quantum dense coding succeeds or not is dependent on the longevity of entanglement states in two-mode or multimode quantum systems. The unavoidable interaction between systems and exterior environments will result in the presence of decoherence, which will degenerate entanglement [8], and weaken the performance of dense coding. In this paper, for the independent and common Bosonic structured environment, we use the physical models provided by R. Vasile et al. in [9, 10] where only the weak system-environment coupling is assumed during the derivation process, and the Markovian approximation and rotating wave approximation are not assumed. Compared with the beamsplitter model, where we only care about the decoherence and entanglement losses for simplification, this model develops a deep and precise understanding on the dynamical features of these phenomena, which can help us to consider a real physical system to control the decoherence and disentanglement and assure that the survival time of entanglement is longer than the time needed for information processing. In this model, we study the effects of environment and derive the analytical expression of channel capacity which qualifies the performance of CV dense coding in terms of both environmental parameters and systemical ones and most of effects are detailed investigated.

This paper is organized as follows. In Section 2, the mutual information of quantum dense coding is derived in the terms of the elements of the evolved covariance matrix of CV two mode squeezed states generated by non-degenerate optical parametric amplifier. In Section 3, we first describe the physical models proposed by R. Vaile et al. [9, 10] in which the two mode Gaussian state evolves. And then based on the analytic expressions of mutual information in Section 2, we give the numerical simulations which show the performance of dense coding depends on the environmental and systemical parameters. The conclusions are drawn in Section 4.

2 Channel Capacity of Quantum Dense Coding Under Bosonic Structured Environments

The executive processes of CV dense coding under independent and common environment are depicted as Fig. 1a and Fig. 1b respectively. Alice and Bob firstly share an Einstein-Podolsky-Rosen(EPR) entangled state approximated by the two-mode squeezed state with characteristic function as

$$\chi_{ab}(\lambda_a, \lambda_b) := Tr\left[\rho_{ab} D\left(\lambda_a\right) D\left(\lambda_b\right)\right] = \exp\left\{-\frac{1}{2}\Lambda^T \sigma_{ab}\Lambda - i\Lambda^T \overline{X}_{ab}\right\}. \quad (1)$$

where $D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is displacement operator and $\hat{a}$ denotes the destroy operator, $\lambda_a = -\frac{i}{\sqrt{2}}(x_a + ip_a)$, $\lambda_b = -\frac{i}{\sqrt{2}}(x_b + ip_b)$ and $\Lambda = (x_a, p_a, x_b, p_b)^T$. Here we define the momentum operator and position one as $X_j = \frac{1}{\sqrt{2}}(\hat{a}_j + \hat{a}_j^\dagger)$, $P_j = -\frac{i}{\sqrt{2}}(\hat{a}_j - \hat{a}_j^\dagger)$ respectively. The covariance matrix σ_{ab} of mode a and mode b reads as

$$\sigma_{\mathbf{ab}} = \begin{pmatrix} A_0 & C_0 \\ D_0 & B_0 \end{pmatrix} \quad (2)$$

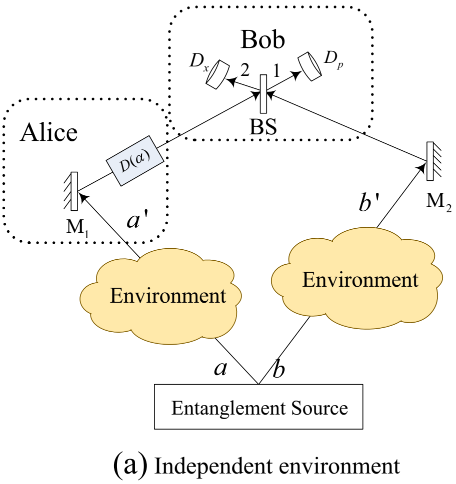


Fig. 1 (Color online) Quantum dense coding under effects of (**a**) independent Bosonic environment and (**b**) common Bosonic environment. Here Entanglement Source denotes device that can generate EPR entangled state, BS denotes beam splitter, and the Arabic numbers denote the modes. D_x and D_p are balanced homodyne detectors. $D(\alpha)$ is displacement operator

where $A_0 = \frac{1}{2}\cosh(2r) \cdot I, B_0 = \frac{1}{2}\cosh(2r) \cdot I, C_0 = D_0 = diag(\frac{1}{2}\sinh(2r), -\frac{1}{2}\sinh(2r))$, r stands for the squeezing parameter, and I is 2×2 identity matrix. And the average values read as

$$\overline{X}_{ab} = Tr\left[\rho_{ab}(X_a, P_a, X_b, P_b)^T\right] = (0, 0, 0, 0)^T. \tag{3}$$

Because of the unavoidable interaction between system and environment, the state ρ_{ab} at time $t = 0$ will evolve into state ρ^t_{ab} at time t according to the open-system master equation [11]. Because of the one-to-one corresponding relationship between quantum state ρ_{ij} and its characteristic function $\chi_{ij}(\lambda_i, \lambda_j)$ reading as

$$\rho_{ij} = \frac{1}{\pi^2}\int d^2\lambda_i d^2\lambda_j \chi_{ij}(\lambda_i, \lambda_j) D_i(-\lambda_i) D_j(-\lambda_j), \tag{4}$$

it is convenient to investigate the evolvement of entangled Gaussian states by studying that of the corresponding characteristic function. For Gaussian evolution such as the interaction between oscillators and Bosonic environments which is used in this paper, the corresponding characteristic function $\chi_{ab}(\lambda_a, \lambda_b)$ still remain its Gaussian form reading as

$$\chi^t_{ab}(\lambda^t_a, \lambda^t_b) = \exp\left\{-\frac{1}{2}\Lambda^T\sigma^t_{ab}\Lambda - i\Lambda^T\overline{X}^t_{ab}\right\}, \tag{5}$$

where the covariance matrix has the most general form which certainly includes the case for interaction between systems and Bosonic structured reservoirs,

$$\sigma^t_{ab} = \begin{pmatrix} A_t & C_t \\ D_t & B_t \end{pmatrix} \tag{6}$$

where $A_t = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix}$, $B_t = \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix}$, $C_t = (D_t)^T = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$. In the next section, the exactly covariance matrix will be given for independent Bosonic reservoir and common one.

Here it's assumed that the average value remains unchanged, because if the average value is unknown to Alice and Bob, they just cannot implement the dense coding protocol, and if the average value is changed they can simply apply additional displacement operation to make them zero. That is to say,

$$\overline{X}^t_{ab} = \overline{X}_{ab} = (0, 0, 0, 0)^T. \tag{7}$$

All above parameters have different expressions for different interaction styles in the different environments. The general form is presented for the convenience of obtaining the general expression of channel capacity of quantum dense coding for Bosonic environments. All above parameters will have the different values for the independent environment (See Fig. 1a) and common environment (Fig. 1b) even both for Bosonic environment.

Alice modulates the information $A_{in} = X_{in} + iP_{in}$ on the quantum mode by applying the displacement operator $D(\alpha_{in})$ on mode A, then she sends it to Bob. Bob measures X_2 and P_1 which are obtained by combining mode A and mode B using beam splitter. Then he could obtain the information modulated by Alice. Here we assume that the two-dimension random variable A_{in} follows Gaussian possibility density function (PDF) $P(x_{in}, p_{in})$ with the average value $E_{in} = (0, 0)^T$ and with the covariance matrix

$$\Sigma_{in} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \tag{8}$$

The characteristic function of the disturbed EPR entangled states of modes 5 and 6 are described by (5–6). Here displacement operation, beam splitter and homodyne measurement are Gaussian operations which transform Gaussian state into another Gaussian state, it is enough to investigate the evolution of both average values and covariance matrix. The displacement operator $D(\alpha_{in})$ changes only the average value of ρ^t_{ab} to $E = (x_{in}, p_{in}, 0, 0)^T$ and keeps the covariance matrix. The beam splitter transform the average values and covariance matrix as following,

$$\overline{X}_{12} = SE = (\cos\theta x_{in}, \cos\theta p_{in}, \sin\theta x_{in}, \sin\theta p_{in})^T \tag{9}$$

$$\sigma_{12} = S\sigma^t_{ab}S^T = \begin{pmatrix} \widetilde{A}_t & \widetilde{C}_t \\ \widetilde{C}^T_t & \widetilde{B}_t \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & c_{11} & c_{12} \\ a_{12} & a_{22} & c_{21} & c_{22} \\ c_{11} & c_{21} & b_{11} & b_{12} \\ c_{12} & c_{22} & b_{12} & b_{22} \end{pmatrix} \tag{10}$$

where $\widetilde{A}_t = \cos^2\theta A_t + \cos\theta\sin\theta(C_t + C^T_t) + \sin^2\theta B_t$, $\widetilde{C}_t = \sin\theta\cos\theta(A_t - B_t) + \sin^2\theta C^T_t - \cos^2\theta C_t$, $\widetilde{B}_t = \cos^2\theta A_t - \cos\theta\sin\theta(C_t + C^T_t) + \sin^2\theta B_t$, $S = \begin{pmatrix} \cos\theta I & \sin\theta I \\ \sin\theta I & -\cos\theta I \end{pmatrix}$ is the transform matrix of beam splitter in phase space, here I is 2×2 identity matrix and θ represents the transmittance t of beam splitter with the relation $t = \cos^2\theta$, $\theta \in [0, \frac{\pi}{2}]$. Thus the characteristic function of quantum state ρ_{12} can be expressed as

$$\chi_{12}(x_1, p_1, x_2, p_2) = \exp\left(-\frac{1}{2}\Lambda^T\sigma_{12}\Lambda - i\Lambda^T\overline{X}_{12}\right) \tag{11}$$

where $\Lambda = (x_1, p_1, x_2, p_2)^T$. The Wigner function $W_{12}(x_1, p_1, x_2, p_2)$ is Fourier transformation of $\chi_{12}(x_1, p_1, x_2, p_2)$. So Wigner function $W_{12}(x_1, p_1, x_2, p_2)$ can be obtain by the following equation,

$$W_{12}(x_1, p_1, x_2, p_2) = \int dx'_1 \int dp'_1 \int dx'_2 \int dp'_2$$
$$\exp[-i(x_1x'_1 + p_1p'_1 + x_2x'_2 + p_2p'_2)]\chi_{12}(x'_1, p'_1, x'_2, p'_2). \tag{12}$$

It is easy to obtain the conditional PDF of the measurement result of X_2 and P_1 according to the following equation,

$$P[(x_2, p_1)|(x_{in}, p_{in})] = \int dp_2 \int dx_1 W_{12}(x_1, p_1, x_2, p_2) \\ = \int dx_2' \int dp_1' \exp[-i(x_2 x_2' + p_1 p_1')]\chi_{12}(0, p_1', x_2', 0). \quad (13)$$

According to (9–11) and (13), we can calculate the PDF $P[(x_2, p_1)|(x_{in}, p_{in})]$ which is Gaussian PDF with the average values $E_{B|A}$ and and covariance matrix $\Sigma_{B|A}$,

$$E_{B|A} = (\sin\theta x_{in}, \cos\theta p_{in})^T \quad (14)$$

$$\Sigma_{B|A} = \begin{pmatrix} b_{11} & c_{21} \\ c_{21} & a_{22} \end{pmatrix} \quad (15)$$

Because $P(x_{in}, p_{in})$ is a Gaussian PDF with covariance matrix given by (8), so it is easy to obtain the joint PDF of the random variables $(x_2, p_1, x_{in}, p_{in})$ according to the following equation,

$$P(x_2, p_1, x_{in}, p_{in}) = P[(x_2, p_1)|(x_{in}, p_{in})]P(x_{in}, p_{in}). \quad (16)$$

According to (14-16), the covariance matrix of PDF $P(x_2, p_1, x_{in}, p_{in})$ read as

$$\Sigma_{AB}^{-1} = \begin{pmatrix} B & C & -B\sin\theta & -C\cos\theta \\ C & A & -C\sin\theta & -A\cos\theta \\ -B\sin\theta & -C\sin\theta & B\sin^2\theta + \frac{1}{\sigma^2} & C\sin\theta\cos\theta \\ -C\cos\theta & -A\cos\theta & C\sin\theta\cos\theta & A\cos^2\theta + \frac{1}{\sigma^2} \end{pmatrix}, \quad (17)$$

where $A = \frac{b_{11}}{b_{11}a_{22}-c_{21}^2}$, $B = \frac{a_{22}}{b_{11}a_{22}-c_{21}^2}$, $C = -\frac{c_{21}}{b_{11}a_{22}-c_{21}^2}$. Thus covariance matrix of PDF $P(x_2, p_1) = \int dx_{in}dp_{in}P(x_2, p_1, x_{in}, p_{in})$ is expressed as

$$\Sigma_B^{-1} = \frac{1}{\sigma^4\cos^2\theta\sin^2\theta(AB - C^2) + (A\cos^2\theta + B\sin^2\theta)\sigma^2 + 1} \\ \times \begin{pmatrix} \sigma^2\cos^2\theta\left(AB - C^2\right) + B & C \\ C & \sigma^2\sin^2\theta\left(AB - C^2\right) + A \end{pmatrix}. \quad (18)$$

The mutual information between Alice and Bob is given by

$$I(A:B) = \int dx_2 \int dp_1 \int dx_{in} \int dp_{in} P\left[(x_2, p_1)|(x_{in}, p_{in})\right] P(x_{in}, p_{in}) \\ \ln\{\frac{P[(x_2, p_1)|(x_{in}, p_{in})]}{P(x_2, p_1)}\} \\ = \frac{1}{2}\ln\{\frac{Det(\Sigma_A)Det(\Sigma_B)}{Det(\Sigma_{AB})}\}. \quad (19)$$

The mutual information in terms of covariance matrix elements is given by substituting (8), (17) and (18) into (19),

$$I(A:B) = \frac{1}{2}\ln\left\{\frac{\sigma^4\cos^2\theta\sin^2\theta + \sigma^2\cos^2\theta b_{11} + \sigma^2\sin^2\theta a_{22} + b_{11}a_{22} - c_{21}^2}{b_{11}a_{22} - c_{21}^2}\right\} \quad (20)$$

If the exact expression of covariance matrix of characteristic function of entangled states ρ_{AB}^t could be obtained, then the dynamical evolution of the mutual information for dense coding could be obtained in terms of environmental parameters, thus the effect of environment on quantum dense coding can be quantitatively analyzed. In Section 3, the physical model for the interaction between oscillators and environment will be introduced, and then give the results of numerical simulations.

3 Numerical Simulations

3.1 Physical Models of Evolution of Two-Mode Squeezed Vacuum States

As is well known, for different interactions in different types of environments, the covariance matrix has different expressions. Here it's assumed that two-mode squeezed vacuum states ρ_{ab} are in Bosonic structured reservoirs either in independent environment or common environment. The dynamical evolution processes of the corresponding characteristic function and their covariance matrix of two-mode squeezed states are detailed investigated according to open-system master equation [11, 12].

Previous work by Ruggero Vasile et al has shown how the two mode squeezed state will evolve in both independent environment [9] and common environment [10]. In their work they give the expression for the characteristic function as a function of t for the time, α for the dimensionless system-reservoir coupling constant, ω_0 for the oscillator frequency and a reservoir described by temperature T characterized by a spectral density $J(\omega)$. In this paper we choose the Ohmic-like spectral distribution as $J_s(\omega) = \omega_c \left(\frac{\omega}{\omega_c}\right)^s \exp\left(-\frac{\omega}{\omega_c}\right)$ [9] in which ω_c stands for the cut-off frequency. When $s > 1$ it is called super-Ohmic, $s = 1$ Ohmic, and $s < 1$ sub-Ohmic. The exact expressions for σ_{ab}^t is tedious and the detail of expression can be found in [9] and [10] so that we just omit these expressions in this paper.

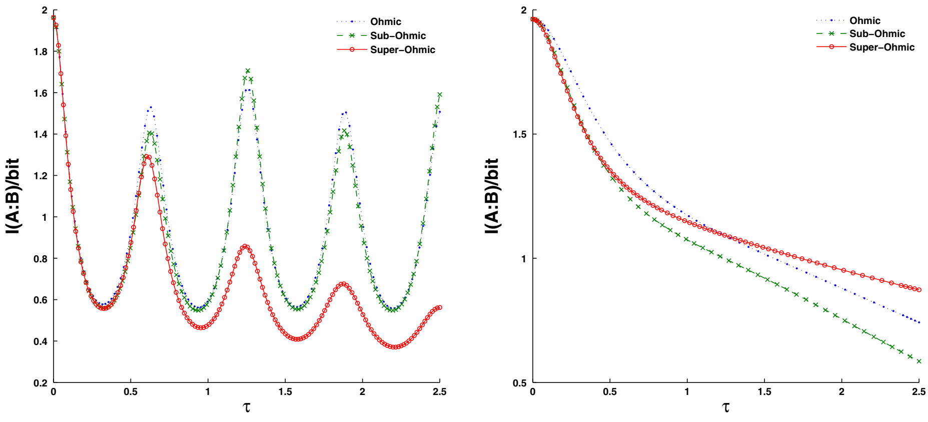
In this section the results of numerical simulation are shown. First the transmittance of the beam splitter is set to $50 : 50 \left(\theta = \frac{\pi}{4}\right)$ and the mutual information varies with time. Second we change θ according to σ_{ab}^t to acquire more mutual information. Though it may not be optimal, we will show that mutual information do increase through such a simple change. Here we fix $r = 0.5$, $\alpha = 0.1$ and $\sigma = 1.5$. It's defined $x = \frac{\omega_c}{\omega_0}$ and $\tau = t \times \omega_0$ for convenience. We also set s to 3, 1 and 0.5 to stand for super-Ohmic, Ohmic and sub-Ohmic spectrum respectively.

3.2 Mutual Information for a Fixed Transmittance

In this section $\theta = \frac{\pi}{4}$ all the time. Figure 2 shows how the mutual information will vary as the time pass by for the independent environment, and Fig. 3 for the common environment. Some interesting property can be concluded from these figures. First the mutual information will oscillate when $x \ll 1$, or it will decrease monotonically when $x \gg 1$. Notice that when $x \ll 1$ the evolution process is non-Markovian evolution and otherwise it is Markovian evolution. So it's assumed that the mutual information is monotonically decreasing function for Markovian interaction and oscillating for non-Markovian. Second, comparing Figs. 2 and 3, it can be seen that mutual information is some larger in common environment than that in independent environment, especially when $x \gg 1$. However whether in independent or in common environment affects dense coding not so much as parameter x.

3.3 Improve the Mutual Information

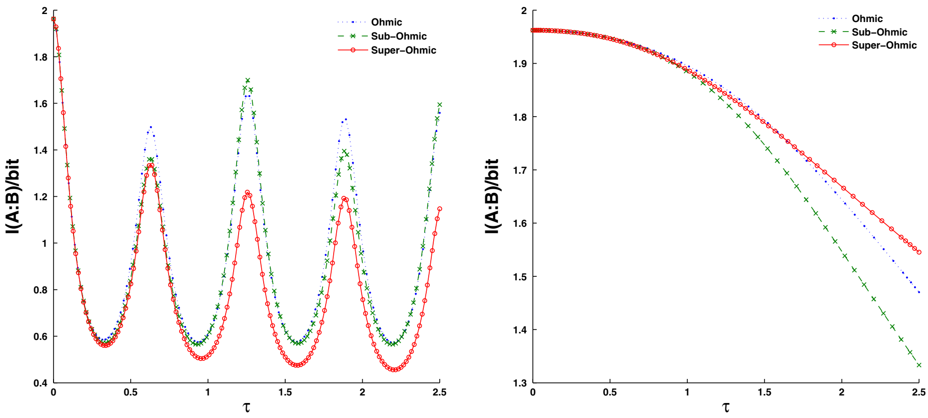
In the previous section, it's shown that generally the mutual information will decrease as the two-mode squeezed state interacted with the some special environment parameters. It is



(a) $x \ll 1$ (b) $x \gg 1$

Fig. 2 Mutual information in independent environment with high temperature, (**a**) $x \ll 1$, namely $x = 0.2$ and (**b**) for $x \gg 1$ namely $x = 10$. In both figure blue line for Ohmic, green line for Sub-Ohmic and red line for Super-Ohmic spectra

unknown that how to achieve the optimal mutual information through a certain entangled state or how much information can be obtained by the receiver Bob. However, in the protocol described in Section 2 the transmittance(θ) of the beam splitter is fixed, which implicate that the mutual information between Alice and Bob could be improved by changing θ according to σ^t_{ab}. In fact, at a certain time σ^t_{ab} is known and thus $I(A:B)$ can be seen as s function of θ. Differentiating (20) one will get the stationary points of $I(A:B)$ and finally calculate θ_{op} to improve the mutual information. Since the derivative and the expression of the stationary point is very tedious and take a long but simple calculation, we just give some numerical



(a) $x \ll 1$ (b) $x \gg 1$

Fig. 3 Mutual information in common environment with high temperature, (**a**) $x \ll 1$, namely $x = 0.2$ and (**b**) for $x \gg 1$ namely $x = 10$. In both figure blue line for Ohmic, green line for Sub-Ohmic and red line for Super-Ohmic spectra

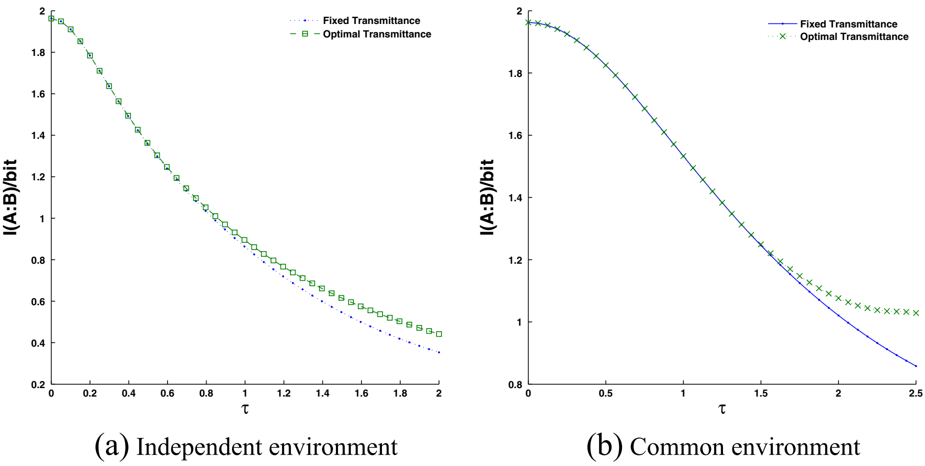


Fig. 4 The improvement of mutual information by changing θ. (**a**)$x = 2.5$, Ohmic and high temperature and (**b**)$x = 3$, Ohmic and ZERO temperature. Blue line for fixed $\theta = \frac{\pi}{4}$ and green for adjusted θ. It can be seen that in both independent and common environment performance of dense coding is improved by choose θ

solution here and show how much information can be increased. In Fig. 4 it is shown that in both independent and common environment the mutual information can increased by changing θ. In fact $\frac{\pi}{4}$ is the optimal chosen of θ at $t = 0$, but probably not the optimal one at other time. Figure 5 illustrates how to chose proper θ to obtain more mutual information.

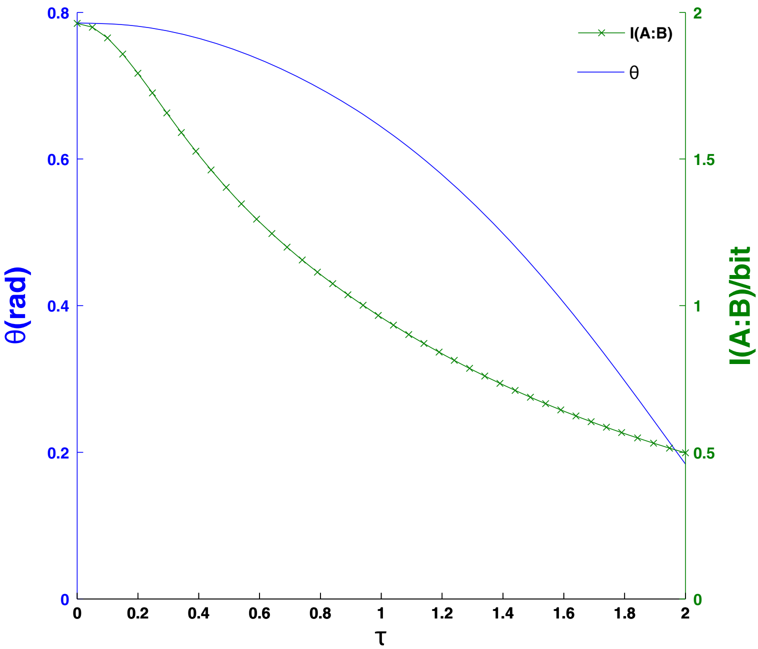


Fig. 5 Blue line shows the suitable θ at different time and the green line shows the maximum mutual information that can achieved only by changing θ. Here $x = 3$, Ohmic and common environment at high temperature

4 Conclusion

In this paper it's analyzed how the continuous dense coding will be affected by the environment. The analytical expression of mutual information between Alice and Bob is given when an arbitary two mode Gaussian state is applied. How the mutual information will change as time pass by is numerically simulated and plotted under various conditions. At last it's suggested to change proper transmittance of beam splitter to improve the performance of dense coding. It's shown that the mutual information can be enhanced by such a simple method.

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