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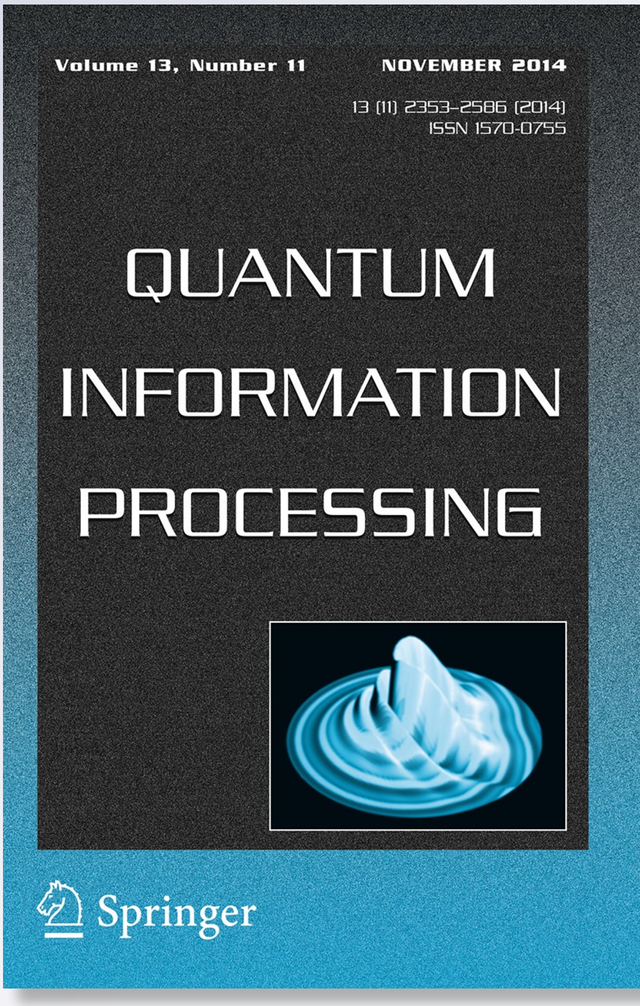

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Abstract We present a dense coding network based on continuous-variable graph state along with its corresponding protocol. A scheme to distill bipartite entanglement between two arbitrary modes in a graph state is provided in order to realize the dense coding network. We also analyze the capacity of network dense coding and provide a method to calculate its maximum mutual information. As an application, we analyze the performance of dense coding in a square lattice graph state network. The result showed that the mutual information of the dense coding is not largely affected by the complexity of the network. We conclude that the performance of dense coding network is very optimistic.

Keywords Quantum · Dense coding · Graph state · Mutual information

1 Introduction

Quantum dense coding is a communication protocol which, making use of an entangled state shared by a sender and a receiver, enables the communication of two classical bits with the transmission of only one quantum bit. The concept of quantum dense coding was first proposed by Bennett and Wiesner in Ref. [1]. Quantum dense coding can be realized via various entanglements [2–6], and it's experimental implementation in continuous variables is also investigated in recent years [7–10].

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Previous researches based on dense coding mainly focused on the point-to-point (P2P) communication, while the ultimate aim of communication, as we all know, is to derive the bipartite communication to a network-like system which involves multiple users. In Ref. [6], the performance of quantum dense coding via a N-qubit entanglement is evaluated. It showed that using maximally entangled qubits is more efficient than pairwise entangled qubits when N is very large. However, no detailed analysis is provided about how to implement dense coding in multiple entanglement states. In Ref. [11], by introducing a special cluster state, the author provide a protocol to implement many-to-one dense coding. However, such cluster state must be restrictedly designed in order to realize the protocol, so it is lack of generalization and application. In this paper, a graph state is used to serves as the entangled resource of dense coding. It enables us to generalize a P2P fixed communication to an any-to-any one which can be applied in future quantum network system.

In our paper, we present a continuous-variable (CV) dense coding network based on an arbitrary graph state and provide its corresponding protocol. Previous researches have already provided some methods to extract entanglement pairs in graph states [12,13]. The restriction of these methods is that the two mode used to distill direct entanglement should be preassigned, in other words, only one bipartite entanglement can be distilled from a given graph state. In contrast, our method in this paper can be used in any two modes in a graph and can distill different bipartite entanglements according to different requirements. This progress is significant because it allows us to use the entanglement resource of graph states more thoroughly and makes it possible to form multipartite quantum communication, not only dense coding. What is more, our protocol is likely to be experimentally realized since the preparation methods of simple graph state is already widely investigated through linear optics [14,15].

This paper is structured as follow. We start by introducing some notions of continuous-variable graph states: we give two methods of the distillation of CV graph states, and proceed by providing a universal dense coding protocol which can be applied on any two modes in a graph state. This protocol is the main result of this paper. In Sect. 3, we analyze the communication capacity, which is characterized by the mutual information of network dense coding in our protocol. A typical application based on square lattice network system is provided in Sect. 4. Finally, we draw our conclusion in Sect. 5.

2 Protocol of dense coding network via graph states

2.1 Basic cognations of CV Graph states

A graph state is a special multiparticle entangled state that can be expressed by a mathematical graph [16]. In a typical graph, each vertex represents a zero-momentum eigenstate($\hat{p} = 0$) and each edge represents a quantum nondemolition (QND) interaction [17]. The QND coupling, characterized by Hamiltonian $H_{ij} = \hbar \chi_{ij} \hat{x}_i \hat{x}_j$, transforms the position and momentum quadratures of two initially isolated mode i and j into the following expressions in the Heisenberg picture [18]:

$$\hat{x}_i^g = \hat{x}_i, \quad \hat{p}_i^g = \hat{p}_i + g_{ij}\hat{x}_j,$$
$$\hat{x}_j^g = \hat{x}_j, \quad \hat{p}_j^g = \hat{p}_j + g_{ji}\hat{x}_i. \tag{1}$$

where $g_{ij} = g_{ji} = -\chi_{ij}t_{ij}$ is the gain of the interaction. χ_{ij} and t_{ij} are the coupling coefficient and the interaction time, respectively. As we can see in Eq. (1), after the QND coupling process, the momentum $\hat{p}_i$ and $\hat{p}_j$ pick up the information of the position $\hat{x}_i$ and $\hat{x}_j$ while the position of the two modes remain unchanged.

Then, for a general N-mode graph state, so long as the adjacency matrix is determined, any modes can be described as

$$\hat{x}_i^g = \hat{x}_i,$$
$$\hat{p}_i^g = \hat{p}_i + \sum_{j=1}^{N} g_{ij}\hat{x}_j, \quad i = 1, 2, \cdots N. \tag{2}$$

Here, $\hat{x}_i = e^{r_i}\hat{x}_i^{(0)}$ and $\hat{p}_i = e^{-r_i}\hat{p}_i^{(0)}$ mean all modes are initially prepared in the squeezed vacuum state. r_i is the squeezing parameter and the superscript (0) denotes the initial vacuum modes. In our protocol, we first assume all modes are infinitely squeezed, i.e., $r_i \rightarrow \infty$ and later consider the nonideal cases where r_i is a finite number so the noise terms are added. In order to express all modes in one equation, we let G denote the adjacency matrix with g_{ij} being the QND coupling coefficient and $\hat{R}$ denote a 2N-dimensional row vector representing the modes' quadratures, i.e., $\hat{R} = (\hat{x}_1 \cdots \hat{x}_N | \hat{p}_1 \cdots \hat{p}_N)$. Then we can rewrite Eq. (2) as

$$\hat{X}^g = \hat{R}\begin{pmatrix} I \\ 0 \end{pmatrix},$$
$$\hat{P}^g = \hat{R}\begin{pmatrix} G \\ I \end{pmatrix}. \tag{3}$$

The above matrix forms would greatly simplify our calculations in later sections.

2.2 Two fundamental operations to distill CV graph states

In order to realize quantum network dense coding and provide corresponding protocol, two primary methods are necessary to distill or, in other words, simplify a certain graph state.

(1) *Disconnection.* This operation is executed by measuring the position of mode i and then performing a displacement operator $\hat{D}(-g_{ij}\hat{x}_i)$ on the momentum quadrature of all neighbors of mode i [12,13,18] ($\hat{D}(s)$ operated on α means $a \rightarrow \alpha + s$).

$$\hat{p}_j^g \rightarrow \hat{p}_j - g_{ij}\hat{x}_i. \tag{4}$$

where mode j is one neighbor of mode i. This operation reduces the scale of graph state from N mode to N-1 mode, so it can be regarded as the inverse operation

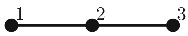

Fig. 1 A linear graph state with three modes

of the QND coupling process. Disconnection is often used to modify a graph state where direct distillation of the bipartite entanglement between two specified modes can not be performed.

(2) *Recasting.* This is the key process of the bipartite entanglement distillation. The process of this operation can be described in the following three steps.
   (i) Measure the momentum quadrature of mode i;
   (ii) Perform $\hat{D}(-\hat{g}_{ij}\hat{p}_i)$ on the position quadrature of one of its neighbor, i.e.,

$$\hat{x}_j^g \rightarrow \hat{x}_j - \hat{g}_{ij}\hat{p}_i. \tag{5}$$

   (iii) Perform an inverse Fourier transformation [12], acting as $\hat{x} \rightarrow -\hat{p}$ and $\hat{p} \rightarrow \hat{x}$ on both the position quadrature and momentum quadrature of mode j. The result is identical to mode j directly entangled with the neighbors of mode i (expect j itself) and carrying the information of mode i while both mode j and the entanglement between mode i and j are destroyed. Here we give an example to illustrate recasting process. Suppose there are only three modes in a graph state and all QND coupling coefficients g_{ij} equal to unity (see Fig. 1).

The initial states are described as:

$$\begin{cases} \hat{x}_1^g = \hat{x}_1, \\ \hat{p}_1^g = \hat{p}_1 + \hat{x}_2, \end{cases} \quad \begin{cases} \hat{x}_2^g = \hat{x}_2, \\ \hat{p}_2^g = \hat{p}_2 + \hat{x}_1 + \hat{x}_3, \end{cases} \quad \begin{cases} \hat{x}_3^g = \hat{x}_3, \\ \hat{p}_3^g = \hat{p}_3 + \hat{x}_2, \end{cases} \tag{6}$$

In order to build connection between mode 1 and mode 3, we measure $\hat{p}_2^g$ and perform $D(-\hat{p}_2^g)$ on $\hat{x}_1^g$,

$$\begin{cases} \hat{x}_1^g = \hat{x}_1 - (\hat{p}_2 + \hat{x}_1 + \hat{x}_3), \\ \hat{p}_1^g = \hat{p}_1 + \hat{x}_2, \end{cases} \tag{7}$$

Then, we perform an inverse Fourier transformation on $\hat{x}_1^g$ and $\hat{p}_1^g$ which results as:

$$\begin{cases} \hat{x}_1^g = \hat{p}_1 + \hat{x}_2, \\ \hat{p}_1^g = \hat{p}_2 + \hat{x}_3, \end{cases} \tag{8}$$

Since $\hat{p}_1 \rightarrow 0$ as we assumed in the head of the section, mode 1 is now directly connected with mode 3.

As we have shown, this operation creates a new entanglement between two initially indirectly connected modes. By performing the recasting operations to all the modes between the source and target mode which we want to form communication, the chain between the two modes would be shortened and finally link the two modes together directly. Since this operation reconstruct the graph by destroying some modes but keeps their information in the remaining modes, just like we melt the metals and recast them into new shapes, we name this operation as recasting.

2.3 Process of setting up bipartite entanglements

Here we provide a method for distilling bipartite entanglements of two arbitrary modes from a graph state network.

We randomly select two modes from the graph, sender Alice and receiver Bob, denote them as mode a $(\hat{x}_a, \hat{p}_a)$ and mode b $(\hat{x}_b,\hat{p}_b)$. Our aim was to obtain the following equation, that Alice's mode and Bob's mode are directly entangled, forming an EPR pair [19]

$$\begin{aligned}&\text{(I)}\ \hat{X}_A - \hat{X}_B = 0,\ \hat{P}_A + \hat{P}_B = 0,\\ &\text{(II)}\ \hat{X}_A + \hat{X}_B = 0,\ \hat{P}_A - \hat{P}_B = 0.\end{aligned} \tag{9}$$

The first equation above is the correlation between position-momentum quadratures for ideal EPR state. The second one is a modified form of the first one by simply applying a Fourier transform on both quadratures. These two equations are equivalent because the entanglement is not affected by local unitary transformation. Here, $(\hat{X}_A, \hat{P}_A)$ represents the final state of Alice after performing a series of displacement operators on the initial mode $(\hat{x}_a, \hat{p}_a)$. These displacements are caused by recasting operations, which is described in previous section. Bob's state does not change, but for clarity, we use $(\hat{X}_B, \hat{P}_B)$ to denote the final mode of Bob; then, in order to implement dense coding, the final two modes should satisfy one of the following two equations.

$$\begin{aligned}\text{(I)}\quad \hat{x}_a^g - u &= \hat{x}_b^g,\\ -\hat{p}_a^g - v &= \hat{p}_b^g.\\ \text{(II)}\ -\hat{x}_a^g - u &= \hat{x}_b^g,\\ \hat{p}_a^g - v &= \hat{p}_b^g.\end{aligned} \tag{10}$$

where u and v are the linear combination of p_i^g, causing by the recasting operations.

$$\begin{aligned}u &= \sum_{i\neq a,b}^{N} \alpha_i p_i^g = \hat{R}\begin{pmatrix} G_{\overline{ab}} \\ I_{\overline{ab}} \end{pmatrix}\alpha,\\ v &= \sum_{i\neq a,b}^{N} \beta_i p_i^g = \hat{R}\begin{pmatrix} G_{\overline{ab}} \\ I_{\overline{ab}} \end{pmatrix}\beta.\end{aligned} \tag{11}$$

Here, $G_{\overline{ab}}$ indicates a $N \times (N-2)$ matrix derived from adjacency matrix G by removing the ath and bth column. $I_{\overline{ab}}$ is similarly defined. α and β are $(N-2)$-dimensional column vectors which would be determined by equations. Notice the condition that when a graph state only has two modes, α and β turn into zero vectors, and the graph state is equivalent to an ideal squeezed two-mode state (EPR pair).

By using the early assumption that the momentum quadrature of all states are infinitely squeezed, we can express Alice and Bob's modes as

$$\begin{aligned} \hat{x}_a^g &= \hat{R}\begin{pmatrix} I_a \\ \forall \end{pmatrix}, \ \hat{p}_a^g = \hat{R}\begin{pmatrix} G_a \\ \forall \end{pmatrix}, \\ \hat{x}_b^g &= \hat{R}\begin{pmatrix} I_b \\ \forall \end{pmatrix}, \ \hat{p}_b^g = \hat{R}\begin{pmatrix} G_b \\ \forall \end{pmatrix}. \end{aligned} \tag{12}$$

Here I_a means the ath column of matrix I. I_b, G_a and G_b have the similar meaning. Now, we can express Eq. (10) in matrix form

$$\begin{aligned} \text{(I)} \ \begin{pmatrix} I_a \\ \forall \end{pmatrix} - \begin{pmatrix} G_{\overline{ab}} \\ I_{\overline{ab}} \end{pmatrix}\alpha &= \begin{pmatrix} I_b \\ \forall \end{pmatrix} \\ -\begin{pmatrix} G_a \\ \forall \end{pmatrix} - \begin{pmatrix} G_{\overline{ab}} \\ I_{\overline{ab}} \end{pmatrix}\beta &= \begin{pmatrix} G_b \\ \forall \end{pmatrix} \\ \text{(II)} \ -\begin{pmatrix} I_a \\ \forall \end{pmatrix} - \begin{pmatrix} G_{\overline{ab}} \\ I_{\overline{ab}} \end{pmatrix}\alpha &= \begin{pmatrix} I_b \\ \forall \end{pmatrix} \\ \begin{pmatrix} G_a \\ \forall \end{pmatrix} - \begin{pmatrix} G_{\overline{ab}} \\ I_{\overline{ab}} \end{pmatrix}\beta &= \begin{pmatrix} G_b \\ \forall \end{pmatrix} \end{aligned} \tag{13}$$

After simplification, we obtain the following equation

$$\begin{aligned} \text{(I)} \ \left(G_{\overline{ab}}| \ - I_a\right)\begin{pmatrix} \alpha \\ \overline{1} \end{pmatrix} &= -I_b, \\ \left(G_{\overline{ab}}| \ + G_a\right)\begin{pmatrix} \beta \\ \overline{1} \end{pmatrix} &= -G_b, \\ \text{(II)} \ \left(G_{\overline{ab}}| \ + I_a\right)\begin{pmatrix} \alpha \\ \overline{1} \end{pmatrix} &= -I_b, \\ \left(G_{\overline{ab}}| \ - G_a\right)\begin{pmatrix} \beta \\ \overline{1} \end{pmatrix} &= -G_b. \end{aligned} \tag{14}$$

Equation (14) is a binary linear equation set, according to the knowledge of linear algebra, we obtain the discriminants

$$\begin{aligned} \text{(I)} \ \operatorname{rank}(G_{\overline{ab}}| - I_a) &= \operatorname{rank}(G_{\overline{ab}}| - I_a| - I_b) \\ \operatorname{rank}(G_{\overline{ab}}| + G_a) &= \operatorname{rank}(G_{\overline{ab}}| - G_a| - G_b) \\ \text{(II)} \operatorname{rank}(G_{\overline{ab}}| + I_a) &= \operatorname{rank}(G_{\overline{ab}}| - I_a| - I_b) \\ \operatorname{rank}(G_{\overline{ab}}| - G_a) &= \operatorname{rank}(G_{\overline{ab}}| - G_a| - G_b) \end{aligned} \tag{15}$$

If the above two equations are satisfied, it means that we can directly distill a bipartite entanglement between mode a and mode b. If not, then some disconnection operations are needed to modify the initial graph state until Eq. (15) is satisfied. By solving vector α and β, we can determine Eq. (10).

When all the modes are infinitely squeezed, u and v approach to zero since they are the linear combinations of $\hat{p}_i$, so the squeezed two-mode state are now established as in Eq. (9). However, since infinitely squeezed states can not be implemented in pratice, noise conditions should be considered into our analysis. Noting that now u and v are actually the total noise of all modes in the graph state (expect the sender and the receiver), we can simply express the noise terms as follow,

$$\sigma_x^2 = \sum_{i \neq a,b}^{N-1} {\alpha_i}^2 e^{-2r_i},$$
$$\sigma_p^2 = \sum_{i \neq a,b}^{N-1} {\beta_i}^2 e^{-2r_i}. \tag{16}$$

where $\langle (\Delta p_i^{(0)})^2 \rangle = 1$. We find that when the conditions are ideal, the bipartite entanglement we distilled is the same as squeezed two-mode state. Nevertheless, (9) also holds for nonideal conditions just with the addition ${\sigma_x}^2$ and ${\sigma_p}^2$.

2.4 Universal protocol of quantum dense coding network

After the discussions in previous sections, we can now provide the universal protocol of quantum network dense coding in CV graph states. This protocol is summarized in five steps.

(i) For a given graph state, determine its adjacency matrix G;
(ii) Choose two modes from the graph, a sender (denoted as a) and a receiver (denoted as b), to perform dense coding;
(iii) Put a, b and G into Eq. (15). If Eq. (15) is satisfied, then use Eq. (14) to solve out vectors α and β;
(iv) If Eq. (15) is not satisfied, then use disconnection operations to discard some modes of the graph state, perform step (iv) repeatedly until Eq. (15) is satisfied;
(v) Find the best solution which can obtain the maximum information capacity in dense coding.

Steps (i)–(iii) are easy to understand since we have analyzed with details in previous subsections. Step (iv) is not always necessary. We can use the methods described in Ref. [12] to get the core graph of a certain graph to satisfy Eq. (14) if needed, but in most cases this is not necessary because in an intentionally built graph state network, the assumption that any two modes can form a bipartite entanglement relationship is usually correct. For step (v), when Eq. (15) is satisfied, the number of solutions of Eq. (14) is either one or infinity. So we need to select the best solution in order to gain the largest mutual information, which is the measurement index of quantum dense coding. The selecting methods will be discussed in Sect. 3.

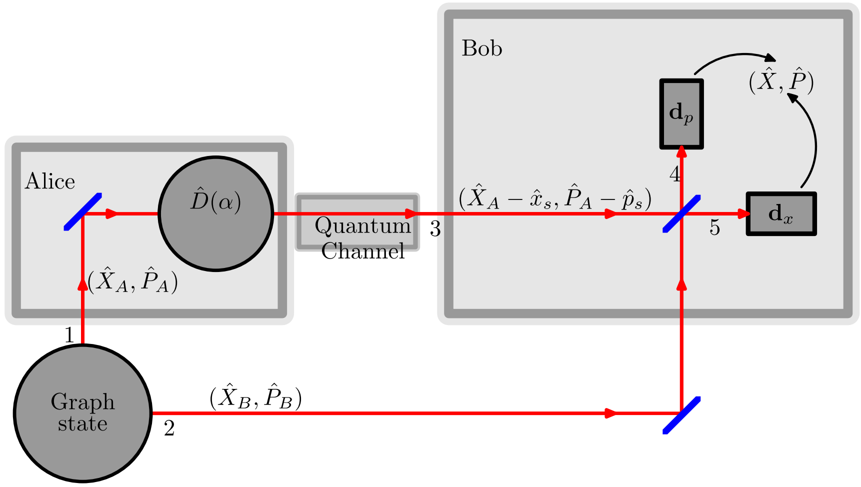

Fig. 2 The topology structure of the dense coding scheme via CV graph state. Beam 1 and beam 2 come from two modes in a graph state which has already formed bipartite entanglement. At Bob's station, homodyne detection is used to form the recovered mode $(\hat{X}, \hat{P})$ (Color figure online)

3 Information capacity analysis

In this section, we discuss the communication quality of dense coding network via CV graph state. Figure 2 shows the topological structure of the implementation of dense coding, which is similar to that in Ref. [2]. We use $\alpha = \langle\hat{x}\rangle + i\langle\hat{p}\rangle$ to indicate the classical signal Alice aims to transfer. This classical information is encoded as a quantum mode $(\hat{x}_s, \hat{p}_s)$, which is shown in Fig. 2. Then, two displacement operators $\hat{D}(-\hat{x}_s)$ and $\hat{D}(-\hat{p}_s)$ are applied to $\hat{X}_A$ and $\hat{P}_A$ separately. This operation encodes the information into beam 1 which now becomes$(\hat{X}_A - \hat{x}_s, \hat{P}_A - \hat{p}_s)$, denoted as beam 3. Then, beam 3 is sent to Bob via a quantum channel. Bob decodes the classical information with the aid of beam 2, another entanglement component generated from the graph state network. The output of Bob, $(\hat{X}, \hat{P})$ would equal to $(\hat{x}_s, \hat{p}_s)$ if all the conditions are ideal.

Now we calculate the mutual information of the communication between Alice and Bob. First we recall the entanglement relationship between Alice and Bob as stated in previous section, the modes of Alice and Bob now satisfy either of the following two equations:

$$\begin{aligned}
&\text{(I)}\ \ \langle(\hat{X}_A - \hat{X}_B)^2\rangle \to \sigma_x^2,\\
&\qquad\langle(\hat{P}_A + \hat{P}_B)^2\rangle \to \sigma_p^2,\\
&\text{(II)}\ \langle(\hat{X}_A - \hat{X}_B)^2\rangle \to \sigma_x^2,\\
&\qquad\langle(\hat{P}_A + \hat{P}_B)^2\rangle \to \sigma_p^2.
\end{aligned} \tag{17}$$

where ${\sigma_x}^2 = \sum_{i\neq a,b}^{N-1} {\alpha_i}^2 e^{-2r_i}$ and ${\sigma_p}^2 = \sum_{i\neq a,b}^{N-1} {\beta_i}^2 e^{-2r_i}$ are the graph state's noise terms which are stated in previous section. Then we identify the Gaussian probability distribution functions(PDF) of both the input signal and output state. We choose the signal α to be distributed as

$$P(\alpha) = \frac{1}{\pi {\sigma_s}^2} \exp(-|\alpha|^2/{\sigma_s}^2). \tag{18}$$

At Bob's receiving station, homodyne detection is used to measure a quantum state β, which is the coupling output of beam 2 and beam 3. According to the homodyne statistics, the distribution of state β is given by

$$P(\beta) = \frac{1}{\pi(\sigma^2 + {\sigma_s}^2)} \exp\left(\frac{-2|\beta^2|}{\sigma^2 + {\sigma_s}^2}\right). \tag{19}$$

with

$$\sigma^2 = {\sigma_x}^2 + {\sigma_p}^2 + e^{-2r}. \tag{20}$$

is the total noise term of the graph state including sender Alice and receiver Bob. Here e^{-2r} is the initial noise of the squeezed two-mode state. The conditional probability of the resulting outcome of ideal homodyne detection is

$$P(\beta|\alpha) = \frac{2}{\pi {\sigma_s}^2} \exp\left(-\frac{2|\beta - \alpha/\sqrt{2}|^2}{{\sigma_s}^2}\right). \tag{21}$$

The two-dimensional mutual information describing the dense coding channel capacity is then given by

$$I = \int d^2\alpha d^2\beta P(\beta|\alpha) P(\alpha) \ln\left(\frac{P(\beta|\alpha)}{P(\beta)}\right). \tag{22}$$

Put Eqs. (18)–(21) into Eq. (22). After simple calculations, we have

$$I(A:B) = \ln\left(1 + \frac{{\sigma_s}^2}{\sigma^2}\right). \tag{23}$$

This equation is quite similar to the Shannon Theory. To simplify our calculations, we suppose that all modes in the graph states are equally squeezed, i.e., $r_i \to r$ for all i. Then, put Eqs. (20) and (16) into Eq. (23), we have

$$I(A:B) = \ln\left\{1 + \frac{{\sigma_s}^2}{e^{-2r}[\sum_{i\neq a,b}^{N-1}(\alpha_i^2 + \beta_i^2) + 1]}\right\}. \tag{24}$$

Clearly, the maximum mutual information is reached when $\sum_{i\neq a,b}^{N-1}(\alpha_i^2+\beta_i^2)$ has the minimum. Recall Eq. (14), of which solution can be expressed as:

$$\alpha = \eta_{\alpha 0} + \sum_{i=1}^{n} c_i \xi_{\alpha_i},$$
$$\beta = \eta_{\beta 0} + \sum_{i=1}^{n} d_i \xi_{\beta_i}, \quad i = 1, 2, \cdots n, \tag{25}$$

here $\eta_{\alpha 0}$, $\eta_{\beta 0}$ are particular roots and ξ_{α_i}, ξ_{β_i} are general roots; c_{α_i},d_{β_i} are undetermined real coefficients and $n = \text{rank}(G_{2:N-1}| - I_1)$. By choosing certain c_i and d_i, we can find the minimum of $\alpha^T\alpha + \beta^T\beta$, thus obtaining the maximum mutual information.

The above step are more of an algorithmic problem but can be easily solved once the adjacency matrix is given. Typical examples will be provided in next section to explain this step in detail. Notice that the mutual information obtained using ideal EPR state for entanglement resource [2] is

$$I_{\text{EPR}} = \ln\left(1 + \frac{\sigma^2}{e^{-2r}}\right). \tag{26}$$

which is just slightly different from what we have attained in Eq. (24). In Sect. 4, we will show by examples that the information capacity is not seriously effected when the scale of network is not so large. This means that we can do dense coding with arbitrary modes in the same network without obvious information loss.

4 Application: Dense coding via square lattice graph states

In practical experiments, by considering the efficiency and practicality, it is important to find out the number of bipartite entanglements that can be directly distilled from a certain network(without extra disconnections process). In this section, we focus on the properties of a $(N \times N)$ square lattice network. We will find out the features of direct bipartite entanglements in the network and then evaluate the mutual information of dense coding in the network.

4.1 The number of direct bipartite entanglements

First, we consider a 3×3 square lattice network (see Fig. 3).

First we select mode 1 to be the sender. By putting $n = 1$ and the adjacency matrix G into Eqs. (14) and (15), we get the solution that $m = 9$, which means mode 9 can be the receiver, i.e., the bipartite entanglement between mode 1 and mode 9 can be directly distilled without any modifications of the network. Similarly, we put $n = 2$ into Eqs. (14) and (15). The solution is $m = 4, 6, 8$. This means that any two modes among 2, 4, 6, 8 can form a bipartite entanglement (see Fig. 4b). The rest can be done in the same method. Finally, we can solve out the number of direct bipartite

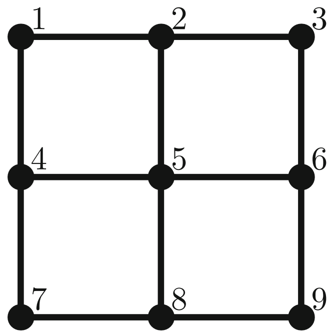

Fig. 3 An 3×3 network, each mode with a serial number indicates a quantum state, here we set the gain of interaction g_{ij} to unity for simplification

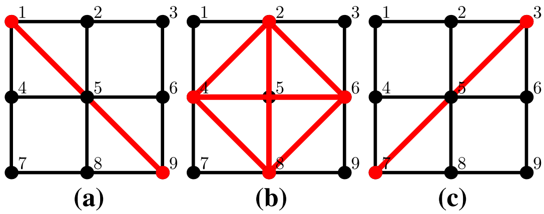

Fig. 4 Bipartite entanglement in 3×3 square lattice network. In each subfigure, any two modes in *red* can be distilled to form a bipartite entanglement without the modification of graph state (mode 1, 9 in **a**; mode 2, 4, 6, 8 in **b**; mode 3, 7 in **c**) (Color figure online)

entanglements that can be distilled from 3×3 network, that is $S_3 = 2 + C_4^2 = 8$. All the conditions are showed in Fig. 4. Notice that for any other two modes in the network such as mode 1 and mode 2, an entanglement can also be distilled out just with a simple modification process [12].

Similarly, in a 4×4 square lattice network, all direct bipartite entanglements are shown in Fig. 5. The number of direct bipartite entanglements is $S_4 = 2 + 2 + 2C_4^2 = 16$.

From what has been stated above, we can generalize the conditions to a $N \times N$ square lattice network by mathematical induction. Any two modes of the rectangles or diagonals with have the same center as the whole network can be distilled directly to form a bipartite entanglement. The number of entanglements is

$$S_{n+1} = \frac{3}{2}n^2 + n - \frac{1}{2}\sin^2\frac{n\pi}{2}, \quad n \geq 1. \tag{27}$$

4.2 Capacity analysis

We first use Fig. 4a as the example to calculate the information capacity of dense coding. Note that the number of modes in this network is 9. We let mode 1 to be the sender (Alice) and mode 9 to be the receiver (Bob). From the analysis above, we know that these two modes can form direct bipartite entanglement without the modification of the network, i.e., Eq. (14) is satisfied. Put $n = 1, m = 9$ into Eq. (15) we have

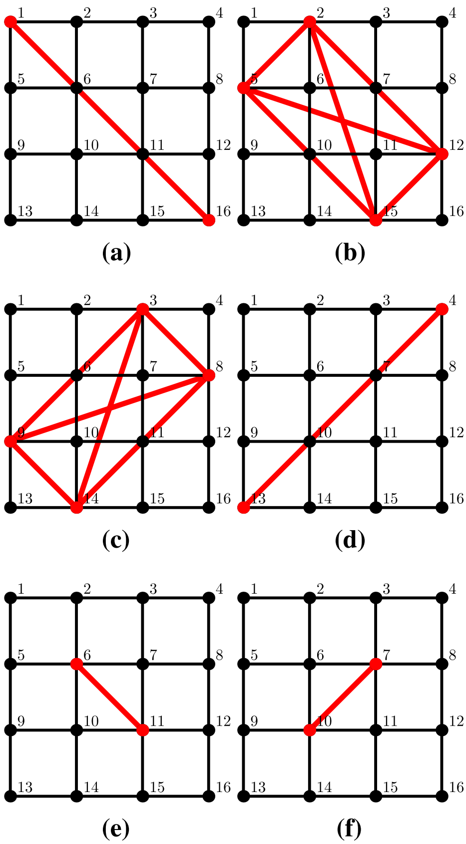

Fig. 5 Bipartite entanglements in 4×4 square lattice network

$$\begin{aligned} \left(\frac{\alpha}{k_1}\right) &= \eta_{\alpha_0} + c_1\xi_{\alpha_1} + c_2\xi_{\alpha_2}, \\ \left(\frac{\beta}{k_2}\right) &= \eta_{\beta_0} + d_1\xi_{\beta_1} + d_2\xi_{\beta_2}. \end{aligned} \tag{28}$$

Here, $\eta_{\alpha*}, \eta_{\beta*}, \xi_{\alpha 1}, \xi_{\alpha 2}$ and$\xi_{\beta 1}, \xi_{\beta 2}$ are all determined column vector, while c_1, c_2 and d_1, d_2 are undetermined coefficients.

Without loss of generality, we suppose all states in the network are equally squeezed and the squeezing parameter $r_i = r, 1 \le i \le N$. Thus Eq. (24) can be rewritten as:

$$I(A:B) = \ln(1 + k{\sigma_s}^2 e^{2r}). \tag{29}$$

where $k = (\sum_{i \neq a,b}^{N-1}(\alpha_i^2 + \beta_i^2) + 1)^{-1}$. c_1, c_2 and d_1, d_2 are easy to be solved out to obtain the minimum of $\sum_{i \neq a,b}^{N-1}(\alpha_i^2 + \beta_i^2)$. After simple calculations, we have $k = 0.2143$. Similarly, the maximum mutual information between other modes which are shown in Fig. 4 only differs in a very small region. By calculation, we have $k = 0.1778$ for mode 2, 4, 6, 8 and $k = 0.1976$ for mode 3, 7. Therefore, we use the information

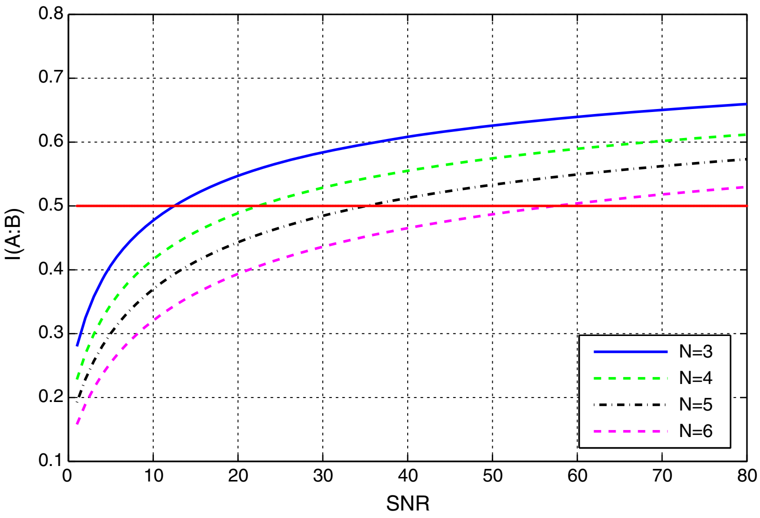

Fig. 6 The *lines* show the relationship of normalized maximum mutual information and SNR. $I(A:B) = 0.5$ is the maximum mutual information for classical communication (highlighted as a *red line* in the graph), which means that all curves above the line are acceptable (Color figure online)

capacity prosperity of the dense coding between mode 1 and mode N^2 to estimate the total performance of a network with given scale.

Then, we consider the maximum mutual information in a larger network. By default, we use mode 1 and mode N^2 for dense coding. We find out that $k = 0.1714$ when the network scale is (4×4), $k = 0.1427$ for (5×5) and $k = 0.1157$ for (6×6). Note that the maximum mutual information of traditional dense coding using EPR state is given in Eq. (26). To estimate the performance of dense coding network via graph state, we normalize $I(A : B)$ with a division of I_{EPR}.

$$I_0(A : B) = \frac{I(A : B)}{I_{\text{EPR}}}. \tag{30}$$

Then, we sketch the normalized maximum mutual information with $\sigma^2 e^{2r}$ as independent variable (Fig. 6). Note that $\sigma^2 e^{2r}$ can be considered as the SNR of the communication system that could be set by Alice and Bob. The graph shows that the performance of our protocol can always beat the classical communication when $\sigma^2 e^{2r}$ is not very small. From the graph, we see that when SNR ≥ 13, the dense coding capacity in 3×3 network can already beats the classical one. And when SNR ≥ 60, which can also be easily realized, dense coding in 6×6 network can also be meaningful.

5 Conclusion

In summary, we have shown how to perform quantum dense coding between arbitrary two modes in a given CV graph state along with the corresponding protocol. A universal method is given to distill an any-to-any bipartite entangled state in a graph state network. This method is not restricted by the structure of the graph or the positions of the two selected modes. With the consideration of noise that caused by finite squeezing, we proved that the performance of dense coding network is very good with only little capacity loss which is in our acceptable scope. By changing the value of SNR, the information capacity can always beat classical communication. Regarding the great facility and efficiency brought by any-to-any communication instead of fixed P2P one, the result we gained is significant.

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